## THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Saturday, February 26. As has now become the established rule the meeting extended through a morning and an afternoon session. This arrangement is not only necessary for the adequate presentation of the large and increasing number of papers offered, but is also generally commended as affording to those present an agreeable opportunity for scientific and social intercourse.

The attendance at the meeting included the following twenty-six members of the Society: Professor E. W. Brown, Professor F. N. Cole, Professor A. M. Ely, Professor Irving Fisher, Professor T. S. Fiske, Mr. G. B. Germann, Professor James Harkness, Mr. P. R. Heyl, Professor Harold Jacoby, Mr. S. A. Joffe, Mr. C. J. Keyser, Professor Pomeroy Ladue, Professor Gustave Legras, Dr. Emory McClintock, Mr. James Maclay, Professor W. F. Osgood, Professor A. W. Phillips, Professor James Pierpont, Professor J. K. Rees, Mr. P. L. Saurel, Professor C. A. Scott, Mr. W. M. Strong, Professor H. D. Thompson, Professor J. M. Van Vleck, Miss E. C. Williams, and Professor R. S. Woodward.

The First Vice-President of the Society, Professor R. S. Woodward, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. Henry Frederick Baker, St. John's College, Cambridge, England; Mr. W. C. Breuke, University of Illinois, Champaign, Ill.; Mr. E. H. Comstock, Cornell University, Ithaca, N. Y.; Professor Edwin Bailey Elliott, Magdalen College, Oxford, England; Professor C. H. Hinton, University of Minnesota, Minneapolis, Minn.; Professor W. E. Stilson, McKendree College, Lebanon, Ill. Six applications for membership were received. The report of the auditing committee was presented and accepted.

The following papers were presented:

(1) Mr. P. L. SAUREL: "On integrating factors."

(2) Professor MAXIME BÔCHER: "The theorems of oscillation of Sturm and Klein (second paper)."

(3) Dr. J. I. HUTCHINSON: "On the tetrahedroid."

(4) Professor W. F. OSGOOD: "A new proof of the existence of a solution of the differential equation $dy/dx = f(x, y)$, the Cauchy-Lipschitz condition not being imposed."

(5) Dr. M. B. PORTER: "On the roots of Bessel's functions."

(6) Professor JAMES PIERPONT: "The early history of the Galoisian theory of equations."

(7) Mr. P. R. HEYL: "The measure of the bluntness of the regular figures in four dimensional space."

(8) Mr. JAMES MACLAY: "Certain double minimal surfaces."

(9) Dr. G. A. MILLER: "On an extension of Sylow's theorem."

(10) Professor H. S. WHITE: "Inflexional lines, triplets, and triangles associated with the plane cubic curve."

(11) Professor W. H. METZLER: "On the excess of the number of combinations in a set which have an even number of inversions over those which have an odd number."

(12) Professor R. S. WOODWARD: "On the significance of the product of the gravitational constant and the mean density of the earth."

In the absence of the authors, the papers of Professor Bôcher, Dr. Hutchinson, Dr. Miller, Professor White, and Professor Metzler were read by title. Dr. Porter's paper was presented by Professor Osgood.

The papers of Dr. Porter and Professor White have already appeared in the March number of the BULLETIN. Those of Mr. Saurel, Dr. Hutchinson, Professor Pierpont and Dr. Miller are contained in the present issue. Those of Professor Bôcher and Mr. Maclay will appear in subsequent numbers. Abstracts of the other papers are given below.

Professor Osgood's paper deals with the fundamental existence theorem of the theory of differential equations. Riemann's definition of the definite integral led to the proof of the existence of a function

$$\varphi(x) = b + \int_a^x f(x)dx$$

which, in case $f(x)$ is continuous—the most general case that concerns us here—satisfies the differential equation $dy/dx = f(x)$ and assumes for $x = a$ the arbitrarily preassigned value $b$. There exists only one such function. Cauchy showed that the differential equation $dy/dx = f(x, y)$, where $f(x, y)$ is continuous in the neighborhood of the point $x = a, y = b$, admits a solution, $y = \varphi(x)$, which assumes the value $b$ when $x = a$, provided $\frac{\partial f(x,y)}{\partial x}$ is finite in the

neighborhood of the point $(a, b)$. The latter condition was generalized by Lipschitz so that it reads: $|f(x, y') - f(x, y)| < k|y' - y|$, $k$ being a constant. In the present paper Professor Osgood shows that by extending what is in substance Riemann's method of proof of the existence of the limit in question to the case of the more general differential equation $dy/dx = f(x, y)$, only the continuity of $f(x, y)$ being assumed, a proof can be obtained that this differential equation has at least one solution taking on the value $b$ when $x = a$. In case more than one such solution exists, any solution $\varphi(x)$ will lie between a certain pair of solutions—the "maximum solution" $\varphi_1(x)$ and the "minimum solution" $\varphi_2(x)$, *i. e.*, $\varphi_2(x) \leqq \varphi(x) \leqq \varphi_1(x)$, and the belt between $\varphi_1(x)$ and $\varphi_2(x)$ will be completely filled with solutions. Finally, a sufficient condition is obtained that $\varphi_1(x)$ and $\varphi_2(x)$ coincide, *i. e.*, that only one solution exists, which is more general than the Cauchy-Lipschitz condition. The condition is as follows; Let $\varphi(u)$ be any continuous function of $u$ such that $\varphi(u) = 0$; $\varphi(u) > 0$ if $u > 0$; $\varphi(-u) = \varphi(u)$; and

$$\lim_{\epsilon=0} \int_{\epsilon}^{u_0} \frac{du}{\varphi(u)} = \infty, \quad u_0 > 0, \epsilon > 0.$$

For example, $\varphi(u) = k|u|$, or $k|u| \log \frac{1}{u|}$, or $k|u| \log \frac{1}{|u|} \log\log \frac{1}{|u|}$, $k$ being a constant. Then the Cauchy-Lipschitz condition, which consists in assuming $\varphi(u) = k|u|$, may be replaced by the following: $|f(x, y) - f(x, y')| < \varphi(y - y')$. The method employed in this paper was geometric, but the final proofs have all been reduced to arithmetic form.

Mr. Heyl obtained numerical values for the magnitude of the angles of the regular figures in four dimensional space. As the bluntness of a regular solid is the measure of the solid angle at its corner, so the bluntness of a regular four-fold figure is the measure of the four-fold angle at its corner. To measure a four-fold angle formed by several solids meeting at a point in four-space, take that point as a center and describe a four-fold sphere of unit radius. The solid sides of the four-fold angle will cut from out the solid boundary of the four-fold sphere a curved solid figure of some sort, whose volume may be taken as the measure of the four-fold angle.

The bluntness of three of the six regular four-fold figures is known from geometrical considerations. Complete bluntness (*i. e.*, the volume of one half the total solid boundary of a four-fold sphere) being $\pi^2 = 9.8696$ we find that the bluntness of the 8-hedroid is 1.234; of the 16-hedroid, 0.8225; and of the 24-hedroid, 2.468. The determination of the bluntness of the three others may be reduced to the problem of finding the volume of a curved isosceles tetrahedron of regular base, when its edges are known. The required formula is obtained by integrating in four-space polar coördinates, and is reduced from a triple integral to a single integral which is numerically evaluated by taking the means of a large number of equally spaced ordinates and multiplying by the difference of the limits of the integral.

This formula is tested upon figures whose bluntness is known, and is found to give correct results. Applying it to the three figures whose bluntness is yet to be determined we obtain the following values, correct to all the figures given: 5-Hedroid, 0.193; 120-Hedroid, 6.28; 600-Hedroid, 5.45.

Professor R. S. Woodward presented an explanation of the significance of the product of the mean density of the earth and the gravitational constant. Referring to a paper "On the gravitational constant and the mean density of the earth," which he had published in the *Astronomical Journal* (No. 424, January, 1898), he stated that he had shown this product to be a quantity of the same kind as the square of an angular velocity, without being able to refer it to any familiar phenomenon. The explanation offered connects this product with the square of the periodic time of an infinitesimal satellite moving without resistance close to the earth along the equator, supposing the earth to be centrobaric. Calling the equatorial radius of the earth $r$, the mean density of the earth $\rho$, the gravitational constant $k$, and the periodic time of the satellite $T$, the attraction of the earth will equal the centrifugal force of the satellite; that is,

$$\tfrac{4}{3}\pi k\rho r^3/r^2 = r(2\pi/T)^2, \text{ whence } k\rho = 3\pi/T^2, \text{ or } k\rho T^2 = 3\pi.$$

In the paper cited above the numerical value of $k\rho$ in C. G. S. units is found to be $k\rho = 36797 \times 10^{-11}/$ (second)$^2$. Hence $T = 5061^s = 1^h\,24^m\,21^s$.

Professor Metzler's paper is in summary as follows: If we are given any combination of $n$ numbers $m$ at a time the combination of the remaining $n-m$ numbers is termed the

complementary with respect to $n$ of the given combination. Suppose now we form the $n_m$ combinations of any $n$ numbers $m$ at a time and consider the set of combinations formed by combining each of these $n_m$ combinations with their complementaries in such a way that the numbers in any combination arranged in their natural order are immediately followed by the numbers in the complementary combination arranged in the same way. The paper then gives an expression for the number of inversions in any combination of this set and also the excess of the number of combinations in the set which have an even number of inversions over those which have an odd number.

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## THE THEOREMS OF OSCILLATION OF STURM AND KLEIN. (FIRST PAPER.)

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of December 29, 1897.)

IN the first volume of *Liouville's Journal* (1836) Sturm has deduced certain properties of the real solutions of linear differential equations of the second order which are of fundamental importance both in pure and in applied mathematics. The opinion has been expressed* that Sturm's work cannot be regarded as rigorous and that other methods must be substituted for his, for instance the method of successive approximations recently employed by Picard for establishing some of these theorems. In one sense it is true that Sturm's work is not rigorous, as hardly any work in analysis done during the first half of the present century shows an appreciation of the difficulties connected with the conception of continuity. The work of Sturm may, however, be made perfectly rigorous without serious trouble and with no real modification of method. In the first two sections of the present paper I have proved such of Sturm's results as are necessary to establish his theorem of oscillation.† In doing this I have departed somewhat from his

* Cf. the first paragraph of Picard's note in the *Comptes Rendus* for February, 1894, and also Klein, Lineare Differentialgleichungen der zweiten Ordnung (lithographed 1894) p. 266: "In der That genügen die Existenzbeweise, wie sie Sturm und Liouville führen, keineswegs den heutigen Anforderungen der Strenge. Man wird verlangen alle die von ihnen gegebenen Entwickelungen in neuer Weise abzuleiten."

† This name is due to Klein.

order of presentation (although even this is a matter of convenience rather than necessity) but his methods are essentially preserved.

I have not thought it desirable to complicate the presentation by preserving at all points the generality of Sturm's memoir.* I have not, however, thought it well to restrict the functions with which we deal to being analytic, for although the proofs would then have appeared simpler this simplicity would have been gained at the expense of their elementary character. On the other hand, I have restricted all the functions with which we deal to being continuous, this being, I suppose, what Sturm intended to do. The questions, some of them very important, which refer to cases in which these functions are discontinuous either within or at an extremity of the intervals in which we consider them, I hope to return to on a future occasion.

Sturm's theorem of oscillation relates to a differential equation which involves a variable parameter. This theorem has been extended by Klein to certain equations involving more than one parameter. We shall confine ourselves in § 3 to Lamé's equation and even then shall consider only the simplest, but, at the same time, the most important case of Klein's theorem of oscillation as Klein himself originally did (*Mathematische Annalen*, vol. 18, 1881). Klein's proof rested entirely on geometric intuition and the form in which I have since presented the proof† is, although the different steps are somewhat more sharply defined, of the same sort.‡ I have given the proof here in what I hope will be found to be a perfectly rigorous analytical form which however rests, as I have briefly indicated in foot notes, upon the same kind of geometric considerations as Klein's original proof.

For the convenience of such readers as are not thoroughly familiar with the methods of exact analysis or with the elements of the theory of linear differential equations, I recapitulate here a few facts of which we shall have to make use.

(A) If throughout and on the boundary of a certain fi-

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* Thus $a$ and $a'$ are here regarded as constant in § 2, instead of being functions of $\lambda$. Cf. also footnote, p. 300.

† Cf. my dissertation: Ueber die Reihenentwickelungen der Potentialtheorie, Göttinger Preisschrift, 1891, and a book published under the same title (to which I shall refer in § 3 as Reihenentwickelungen) in 1894 by Teubner. See also Klein: Lineare Differentialgleichungen der zweiten Ordnung, 1894.

‡ An analytic form of proof is also given by Pockels in which, however, questions of continuity are disregarded. See p. 118 of his book: Ueber die Differentialgleichung $\Delta u + k^2 u = 0$. Leipzig, Teubner, 1891.

nite region $f(x, y, z, \ldots)$ is a continuous function of $(x, y, z, \ldots)$ it will be *uniformly continuous* there.* From this it follows immediately that if when $a \leqq x \leqq b$ and $c \leqq y \leqq d$ $f(x, y)$ is a continuous function of $(x, y)$ and if when $y = y_0$ $(c \leqq y_0 \leqq d)$ $f(x, y)$ does not vanish when $a \leqq x \leqq b$ then it is possible to find a positive quantity $\varepsilon$ so small that when $|y - y_0| < \varepsilon$ (and also $c \leqq y \leqq d$) $f(x, y)$ does not vanish when $a \leqq x \leqq b$.

(B) If when $a \leqq x \leqq b$, $\lambda_1 \leqq \lambda \leqq \lambda_2$, $\mu_1 \leqq \mu \leqq \mu_2$ $f(x, \lambda, \mu)$ is a single valued continuous function of $(x, \lambda, \mu)$ then in the same intervals $\int_a^x f(x, \lambda, \mu)\, dx$ will be a continuous function of $(x, \lambda, \mu)$.†

(C) If when $a \leqq x \leqq b$ $p(x)$ and $q(x)$ are single valued continuous functions of $x$ one and only one function $y(x)$ exists, which at every point of the interval $ab$ satisfies the differential equation:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

(and which, therefore, has continuous first and second derivatives throughout $ab$) and which at the point $x_0 (a \leqq x_0 \leqq b)$ satisfies the relations $y(x_0) = a$, $y'(x_0) = a'$ ($a$ and $a'$ arbitrary real constants).‡

In particular if $a = a' = 0$, $y$ is identically zero.

(D) If the functions $p$ and $q$ in (C) involve a parameter $\lambda$ in such a way that when $a \leqq x \leqq b$ and $\lambda_2 \leqq \lambda \leqq \lambda_1$ they are continuous functions of $(x, \lambda)$ then the function $y$ determined in (C) will be in the same region a continuous function of $(x, \lambda)$. §

(E) The function $y$ determined in (C) cannot have an infinite number of roots in the interval $ab$ unless $a = a' = 0$. For if it had, these roots would have at least one limiting point $x_1 (a \leqq x_1 \leqq b)$ and since $y$ is continuous at $x_1$ $y(x_1) = 0$. Moreover by Rolle's theorem there will be in the neighbor-

*Cf. Jordan: Cours d'Analyse, 2d edition, vol. I, p. 48; Harkness and Morley: Theory of Functions, §64; or Picard: Traité d'Analyse, vol. I, p. 3 and p. 90 (where, however, the term *uniform continuity* is not mentioned). In the last two works the important condition that the boundary must belong to the region of continuity is not explicitly stated.

†I am not able to give a reference to a proof of this theorem. Such a proof may be given in a manner precisely similar to that in which it is ordinarily proved that the above integral is a continuous function of $(\lambda, \mu)$ and also of $x$. See Harnack's Calculus §146, VIII, and §151.

‡Cf. Picard: Traité d'Analyse vol. II., Chap. 11; or Jordan: Cours d'Analyse, vol. III, p. 88.

§This follows immediately from the method of successive approximations. See for a special case Picard: Traité d'Analyse, vol. III, p. 93.

hood of $x_1$ an infinite number of points at which $y' = 0$, and therefore since $y'(x)$ is continuous $y'(x_1) = 0$.* From the remark at the end of (C) we see that $y$ and $y'$ cannot both vanish at $x_1$.

(F) If $x_0$ is a root of the function $y$ determined in (C) $(a \leqq x_0 \leqq b)$, then for sufficiently small values of $\varepsilon$ $y(x_0 + \varepsilon)$ and $y(x_0 - \varepsilon)$ have opposite signs since $y'(x_0) \neq 0$; and if $x_0'$, $x_0''$ are two successive roots, $y'(x_0')$ and $y'(x_0'')$ have opposite signs.

(G) By the change of dependent variable:

$$\bar{y} = e^{\frac{1}{2}\int p\,dx} \cdot y$$

or by the change of independent variable:†

$$t = \int e^{-\int p\,dx}\,dx$$

the differential equation in (C) may be reduced to the binomial form:

$$\frac{d^2y}{dx^2} = \varphi(x) \cdot y.$$

We shall confine our attention to equations of this form in the following §§ 1, 2.

## §1. *The Comparison of Corresponding Solutions of Two Differential Equations.*

We will begin by establishing the following fundamental *Theorem of Comparison:*

I. *If in the differential equations:*

$$\frac{d^2y}{dx^2} = \varphi_1(x) \cdot y \tag{1}$$

$$\frac{d^2y}{dx^2} = \varphi_2(x) \cdot y \tag{2}$$

*$\varphi_1(x)$ and $\varphi_2(x)$ are single valued continuous functions when $a \leqq x \leqq b$ and if throughout this interval $\varphi_1(x) \geqq \varphi_2(x)$ (the equal-*

* This follows more simply still from the fact that $[y(x_1 + \Delta x) - y(x_1)] / \Delta x$ keeps vanishing as $\Delta x$ approaches zero. The proof given in the text is however more convenient than the one just indicated in the case of equations of higher order tha ı the second.

† It should be noticed that since $dt / dx > 0$, an interval $ab$ of the $x$-axis throughout which $p$ is continuous corresponds in a one to one manner to an interval of the $t$-axis, and that any set of points in the first interval follow each other in the same order as the corre-ponding points in the second interval.

*ity sign not holding for all values of $x$ in the neighborhood of $a$); if moreover $y_1$ and $y_2$ denote solutions of* (1) *and* (2) *respectively which satisfy the conditions $y_1(a) = y_2(a) = a$, $y_1'(a) = y_2'(a) = a'$ and if $y_1$ has $n$ roots $x_1, x_2, \cdots, x_n$ such that $a < x_1 < x_2 \cdots < x_n \leqq b$; then $y_2$ will have at least $n$ roots between $a$ and $b$ and the $i$th $(i = 1, 2, \cdots, n)$ of these roots measured from $a$ is less than $x_i$.*

Let us consider the differential equation:

$$\frac{d^2y}{dx^2} = \left[\varphi_2(x) + \lambda\left(\varphi_1(x) - \varphi_2(x)\right)\right] y. \tag{3}$$

This equation reduces when $\lambda = 1$ to (1) and when $\lambda = 0$ to (2). Let $y(x, \lambda)$ be the solution of (3) which satisfies the conditions:

$$y(a, \lambda) = a, \quad y'(a, \lambda) = a',$$

so that $y(x, 1) = y_1(x)$ and $y(x, 0) = y_2(x)$.

For what follows it is important to see that we can always find a constant $a'$ which exceeds $a$ by so little that when $a < x \leqq a'$ and $0 \leqq \lambda \leqq 1$ $y(x, \lambda) \neq 0$. This is at once obvious from (A) when $a$ is not zero. When $a = 0$ we get our result in a similar manner by considering $y'(x, \lambda)$.

Now consider the differential equations satisfied by $y(x, \lambda_1)$ and $y(x, \lambda_2)$ respectively. Multiply the first of these equations by $y(x, \lambda_2)$ the second by $y(x, \lambda_1)$ and subtract. This gives:

$$y(x, \lambda_2)\, y''(x, \lambda_1) - y(x, \lambda_1)\, y''(x, \lambda_2) =$$

$$(\lambda_1 - \lambda_2)\left[\varphi_1(x) - \varphi_2(x)\right] y(x, \lambda_1)\, y(x, \lambda_2).$$

Let us integrate this equation from $x = a$ to $x = x$:

$$y(x, \lambda_2)\, y'(x, \lambda_1) - y(x, \lambda_1)\, y'(x, \lambda_2) =$$

$$(\lambda_1 - \lambda_2) \int_a^x \left[\varphi_1(x) - \varphi_2(x)\right] y(x, \lambda_1)\, y(x, \lambda_2)\, dx. \tag{4}$$

The integral which on the right hand side of this equation is multiplied into $\lambda_1 - \lambda_2$ is by (B) a continuous function of $(x, \lambda_1, \lambda_2)$ when $a' \leqq x \leqq b$, $0 \leqq \lambda_1 \leqq 1$, $0 \leqq \lambda_2 \leqq 1$. Moreover when $\lambda_1 = \lambda_2$ it is positive within the limits above indicated (it cannot be zero since it is at least as great as the integral from $a$ to $a'$). Accordingly, it follows from (A) that it is possible to find a positive quantity $\varepsilon$ independent of $x$, $\lambda_1$, and $\lambda_2$ and so small that when $|\lambda_1 - \lambda_2| < \varepsilon$ the above integral is positive throughout the region mentioned. Let us then insert between the values $\lambda = 1$ and $\lambda = 0$ a number of other

values $k_1, k_2, \cdots, k_m$ in such a way that all the differences $1 - k_1, k_1 - k_2, k_2 - k_3, \cdots, k_m - 0$ are less than $\varepsilon$. Then if we let $\lambda_1$ and $\lambda_2$ be two successive values taken from the set $1, k_1, k_2, \cdots, k_m, 0$ the integral just discussed will be positive when $a' \leqq x \leqq b$.

Let us begin by letting $\lambda_1 = 1$, $\lambda_2 = k_1$, $x = x_i$. Then since $y(x_i, 1) = 0$ the second term on the left hand side of (4) drops out, while the right hand side is positive. We see then that $y(x_i, k_1)$ and $y'(x_i, 1)$ have the same sign. In the same way $y(x_{i-1}, k_1)$ and $y'(x_{i-1}, 1)$ have the same sign. But $y'(x_i, 1)$ and $y'(x_{i-1}, 1)$ have opposite signs (see (F)), and therefore $y(x_i, k_1)$ and $y(x_{i-1}, k_1)$ have opposite signs. Accordingly $y(x, k_1)$ has at least one root between $x_{i-1}$ and $x_i$. By similar reasoning $y(x, k_1)$ has at least one root between $a'$ and $x_1$ (since $y(a', 1)$ and $y(a', k_1)$ have the same sign while $y'(x_1, 1)$ and therefore $y(x_1, k_1)$ has the opposite sign from them).

There cannot be more than one root in any one of the intervals just considered. For if there were let $\mu$ and $\nu$ be two successive roots of $y(x, k_1)$ between which no root of $y(x, 1)$ lies. We see then by using (4) as before that $y(\mu, 1)$ and $y(\nu, 1)$ have respectively opposite signs from $y'(\mu, k_1)$ and $y'(\nu, k_1)$ while these last two quantities have opposite signs from each other. $y(\mu, 1)$ and $y(\nu, 1)$ must then have opposite signs. But this is impossible since $y(x, 1)$ has no root between $\mu$ and $\nu$.

Our theorem now follows at once. For we have just proved that $y(x, k_1)$ has at least $n$ roots between $a$ and $b$ and that if $x_i'$ $(i = 1, 2, \cdots, n)$ denotes the $i$th of these roots from $a$ $x_i' < x_i$. In precisely the same way it follows that $y(x, k_2)$ has at least $n$ roots between $a$ and $b$ and that if

$$x_i''(i = 1, 2, \cdots, n)$$

denotes the $i$th of these roots from $a$ $x_i'' < x_i' < x_i$. Proceeding in this way we finally get the theorem concerning the roots of $y(x, 0)$ above stated.

To the theorem just established may be added a second *Theorem of Comparison:*

II. *The notation and conditions* being the same as in Theorem I, if neither $y_1(b)$ nor $y_2(b)$ is zero and if $y_1(x)$ and $y_2(x)$ each has just $n$ roots between $a$ and $b$ ($n$ may be zero), then:*

$$y_1'(b) / y_1(b) > y_2'(b) / y_2(b).$$

* It is easy to show that the restriction we made in Theorem I, that the equality sign in the relation $\phi_1(x) \geqq \phi_2(x)$ shall not hold for all points in the neighborhood of $a$ may here be replaced by the condition that it must not hold for all points of the interval $ab$.

In the following proof we will use the same notation as in the proof of Theorem I.

In the first place it is clear that each of the functions $y(x, k_1)$, $y(x, k_2)$, $\cdots$, $y(x, k_m)$ has just $n$ roots between $a$ and $b$ and does not vanish when $x = b$. For by Theorem I it must have at least $n$ roots greater than $a$ and less than $b$ since this is true of $y(x, 1)$; and it cannot have more than $n$ such roots nor can it vanish when $x = b$ since then by Theorem I $y(x, 0)$ would have at least $n + 1$ roots between $a$ and $b$. It follows then from (F) that the functions $y(x, 1)$, $y(x, k_1)$, $\cdots$, $y(x, k_m)$, $y(x, 0)$ have changed sign the same number of times in passing from $x = a'$ to $x = b$, and since they all have the same sign when $x = a'$ they must all have the same sign when $x = b$.

The method we will use to prove our theorem is to establish the continued inequality:

$$\frac{y'(b, 1)}{y(b, 1)} > \frac{y'(b, k_1)}{y(b, k_1)} > \frac{y'(b, k_2)}{y(b, k_2)} > \cdots > \frac{y'(b, 0)}{y(b, 0)}.$$

The first step of this inequality will be established if we can prove that the difference between the two first terms:

$$\frac{y(b, k_1)\, y'(b, 1) - y(b, 1)\, y'(b, k_1)}{y(b, 1)\, y(b, k_1)}$$

is positive. We have just seen that $y(b, 1)$ and $y(b, k_1)$ have the same sign, so that the denominator is positive while the numerator is at once seen to be positive when we let $\lambda_1 = 1$, $\lambda_2 = k_1$ in (4). In precisely the same way we prove the other inequalities above written.

The two preceding theorems enable us to compare an equation we wish to investigate with one about which something is known. The equation most frequently taken as standard of comparison is:

$$\frac{d^2y}{dx^2} = cy \tag{5}$$

where $c$ is a constant. We give now some theorems which can be obtained by such a comparison and which we shall need in the following sections.

When $c$ is positive the solution of (5) which satisfies the conditions $y(a) = a$, $y'(a) = a'$ is:

$$y = \frac{a\sqrt{c} + a'}{2\sqrt{c}} e^{\sqrt{c}(x-a)} + \frac{a\sqrt{c} - a'}{2\sqrt{c}} e^{-\sqrt{c}(x-a)}.$$

It is easily seen that when $a = 0$ $y$ vanishes only when $x = a$, and that when $a \neq 0$ and $c \geqq \frac{a'^2}{a^2}$ $y$ does not vanish at all. We may then deduce the following proposition from Theorem I:

III. *If $\varphi(x)$ is single valued and continuous when $a \leqq x \leqq b$ and if $y$ is the solution of:*

$$\frac{d^2y}{dx^2} = \varphi(x)\, y$$

*which satisfies the conditions $y(a) = a$, $y'(a) = a'$ then, provided that throughout the interval $ab$ $\varphi(x) \geqq a'^2/a^2$ (or if $a = 0$ provided that $\varphi(x) \geqq 0$), $y(x)$ will not vanish when $a < x \leqq b$.*

By applying Theorem II we get:

IV. *If $y$ is determined as in Theorem III and if $\varphi(x) \geqq c$ (the equality sign not holding for all values of $x$ in the neighborhood of $a$) then provided that $c \geqq a'^2/a^2$ (or if $a = 0$ provided that $c \geqq 0$) we have when $a < x \leqq b$:*

$$\frac{y'(x)}{y(x)} > \sqrt{c}\,\frac{(a\sqrt{c} + a')e^{\sqrt{c}(x-a)} - (a\sqrt{c} - a')e^{-\sqrt{c}(x-a)}}{(a\sqrt{c} + a')e^{\sqrt{c}(x-a)} + (a\sqrt{c} - a')e^{-\sqrt{c}(x-a)}}.$$

We may add to Theorems III and IV as a corollary whose truth may also be seen directly:

*If $\varphi(x) \geqq 0$ and either $a = 0$ or $a' = 0$, then when $a < x \leqq b$ $y(x)$ does not vanish and $y'/y > 0$.*

If on the other hand $c$ is negative, any solution of (5) may be written in the form:

$$y = C_1 \cos \sqrt{-c}\,x + C_2 \sin \sqrt{-c}\,x,$$

and since this function has an infinite number of roots situated at intervals of $\pi / \sqrt{-c}$ we get the theorem:

V. *If $y$ is determined as before and if throughout the interval $ab$:*

$$\varphi(x) < -\frac{n^2\pi^2}{(b-a)^2}$$

*then $y$ will have at least $n$ roots in the interval $ab$.*

## §2. *On the Solutions of Differential Equations involving one Parameter.*

We have been dealing so far with differential equations

which do not involve parameters.* We come now to some theorems concerning the solutions of the equation:

$$\frac{d^2y}{dx^2} = \varphi(x, \lambda)y \tag{6}$$

in which we will suppose that $\varphi$ is a continuous function of $(x, \lambda)$ when $a \leqq x \leqq b$ and $\lambda_1 \leqq \lambda \leqq \lambda_2$. Let us denote by $y(x, \lambda)$ the solution of (6) which satisfies the conditions $y(a, \lambda) = \alpha$, $y'(a, \lambda) = \alpha'$ ($\alpha$ and $\alpha'$ independent of $\lambda$). We get then the theorem:

VI. *If when $a \leqq x \leqq b$ and $\lambda_1 \leqq \lambda' > \lambda'' \leqq \lambda_2$ $\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$ (the equality sign not holding for all values of $x$ in the neighborhood of $a$) and if $y(x, \lambda_1)$ has $n$ roots in the interval $a < x \leqq b$ then $y(x, \lambda)$ $(\lambda_1 \leqq \lambda \leqq \lambda_2)$ will have at least $n$ roots in this interval. Moreover, if we call these roots (or if there are more than $n$ the first $n$ of them) $x_1(\lambda), x_2(\lambda), \cdots, x_n(\lambda)$ where $a < x_1 < x_2 < \cdots < x_n \leqq b$ then $x_i(\lambda)$ is a single valued and continuous function of $\lambda$ when $\lambda_1 \leqq \lambda \leqq \lambda_2$.*

Everything here stated follows at once from Theorem I except the continuity of $x_i(\lambda)$. In order to establish this continuity it is sufficient (and necessary) to show that, $\lambda'$ being any value of $\lambda$ in the interval $\lambda_1\lambda_2$, no matter how small a positive quantity $\delta$ we may take it is possible to find a positive quantity $\varepsilon$ so small that when

$$|\lambda - \lambda'| < \varepsilon, \ |x_i(\lambda) - x_i(\lambda')| < \delta \ (i = 1, 2, \cdots, n).$$

To show this let $\delta$ be taken so small that in the intervals

$$|x_i(\lambda') - x| \leqq \delta \ (i = 1, 2, \cdots, n) \quad y'(x, \lambda') \neq 0.$$

These intervals we will denote by $I_1, I_2, \cdots, I_n$. The possibility of this choice of $\delta$ is obvious from the fact that

$$y'(x_i(\lambda'), \lambda') \neq 0$$

and from the continuity of $y'(x, \lambda')$. Let us now notice that just as on p. 299 it is possible to find a quantity $a'$ such that when $a < x \leqq a'$ and $\lambda_1 \leqq \lambda \leqq \lambda_2$ $y(x, \lambda) \neq 0$; and consider the intervals $a' \leqq x \leqq x_1(\lambda') - \delta$, $x_1(\lambda') + \delta \leqq x \leqq x_2(\lambda') - \delta$, $\cdots$, $x_{n-1}(\lambda') + \delta \leqq x \leqq x_n(\lambda') - \delta$ which we will call $J_1, J_2, \cdots, J_n$. Since $y(x, \lambda')$ does not vanish in any of these intervals while $y'(x, \lambda')$ does not vanish in any of the intervals $I_1, \cdots, I_n$ it is possible, (see (A).) to choose $\varepsilon$ so small that when

* The introduction which we found it convenient to make of a parameter $\lambda$ in the proof of Theorems I and II does not affect this statement.

$|\lambda - \lambda'| < \varepsilon$ $y(x, \lambda)$ does not vanish in any of the intervals $J$ nor at the point $x_n(\lambda') + \delta$ and $y'(x, \lambda)$ does not vanish in any of the intervals $I$. This being done it is clear that when $|\lambda - \lambda'| < \varepsilon$ $y(x,\lambda)$ has one and only one root in each of the intervals $I$. For if it had more than one root in any of these intervals $y'(x,\lambda)$ would have, by Rolle's theorem, to vanish in this interval. And it must have at least one root in each interval $I_i$ since $y(x, \lambda')$ and therefore $y(x, \lambda)$ has opposite signs in the intervals $J_i$ and $J_{i+1}$ (or when $i = n$ in the interval $J_n$ and at the point $x_n(\lambda') + \delta$). Moreover since $y(x, \lambda)$ has no roots in the intervals $J$ or in the interval $a < x \leqq a'$ it follows that the root which we have just seen lies in the interval $I_i$ is the root $x_i(\lambda)$. Our theorem is thus proved.*

VII. *If in (6) when* $a \leqq x \leqq b$ *and* $\lambda_1 \geqq \lambda' > \lambda'' \geqq \lambda_2$

$$\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$$

*(the equality sign not holding for all points in the neighborhood of a)* † *and if* $y(x, \lambda_1)$ *has just n roots (n may be zero) when* $a < x \leqq b$ *and* $y(x, \lambda_2)$ *has just* $n + m$ *roots in this interval then there will be one and only one value of* $\lambda$ $(\lambda_1 > \lambda \geqq \lambda_2)$ *for which* $y(x, \lambda)$ *vanishes when* $x = b$ *and has just* $n + k$ *roots* $(0 < k < m)$ *between a and b.*

In order to prove this let us define the function $\varphi(x, \lambda)$ when $x > b$ by the formula $\varphi(x, \lambda) = \varphi(b,\lambda) - (x - b)$. The function as thus defined is a continuous function of $(x,\lambda)$ when $\lambda_1 \geqq \lambda \geqq \lambda_2$ and $a \leqq x$; and throughout this whole region it satisfies the relation $\varphi(x,\lambda') \geqq \varphi(x,\lambda'')$ when $\lambda' > \lambda''$. Moreover when $x$ increases beyond $b$ $\varphi(x,\lambda)$ decreases indefinitely so that by Theorem V $y(x,\lambda)$ has for all values of $\lambda(\lambda_1 \geqq \lambda \geqq \lambda_2)$ an infinite number of roots greater than $b$. Let us call the first $m + n$ roots of $y(x,\lambda)$ which are greater than $a$

$$x_1(\lambda), x_2(\lambda), \cdots, x_{m+n}(\lambda).$$

Then by Theorem VI $x_{m+k}(\lambda)$ is a single valued and continuous function of $\lambda$ which constantly decreases as $\lambda$ varies from $\lambda_1$ to $\lambda_2$. Therefore since $x_{m+k}(\lambda_1) > b$ and $x_{m+k}(\lambda_2) \leqq b$ there must be one and only one value of $\lambda$ $(\lambda_1 > \lambda \geqq \lambda_2)$ for which $x_{m+k}(\lambda) = b$.

We come now to Sturm's theorem of oscillation :

* Compare the proof just given with the treatment of implicit functions in Jordan's Cours d'Analyse 2d edition, vol. I., p. 80.

† Here, as in the proof of Theorem II, all that need be required is that the equality sign shall not hold for all values of $x$ in the interval $ab$. This remark applies to the next theorem also.

VIII. *If when* $a \leqq x \leqq b$ *and* $L > \lambda > l$ $\varphi(x, \lambda)$ *is a single valued continuous function of* $(x, \lambda)$ *and if when* $a \leqq x \leqq b$ *and* $L > \lambda' > \lambda'' > l$ $\varphi(x, \lambda') \geqq \varphi(x, \lambda'')$ *(the equality sign not holding for all values of* $x$ *in the neighborhood of* $a$*);* and finally if no matter how large the quantity* $M$ *and how small the quantity* $m$ *may be chosen it is possible to take* $\lambda$ *on the one hand so near to* $L$ *that for all values of* $x$ *in the interval* $ab$ $\varphi(x, \lambda) > M$*, and on the other hand so near to* $l$ *that for all valves of* $x$ *in the interval* $ab$ $\varphi(x, \lambda) < m$*; then there is one and only one value of* $\lambda$ $(L > \lambda > l)$ *for which the differential equation:*

$$\frac{d^2y}{dx^2} = \varphi(x, \lambda)\, y$$

*has a solution which has* $n$ *roots greater than* $a$ *and less than* $b$ *and which satisfies the conditions:*

$$\alpha' y(a) - \alpha y'(a) = 0, \tag{1}$$

$$\beta' y(b) - \beta y'(b) = 0. \tag{2}$$

Here $n$ is an arbitrarily chosen integer positive or zero, and $\alpha$, $\alpha'$, $\beta$, $\beta'$ are arbitrarily chosen real quantities restricted merely by the fact that $\alpha$ and $\alpha'$ must not both be zero and $\beta$ and $\beta'$ must not both be zero. Moreover, it is not necessary to require that $L$ and $l$ should be finite; we may have $L = +\infty$ or $l = -\infty$ or both of these cases may occur at once.

In the special case in which $\beta = 0$ the truth of this theorem follows at once from Theorems III, V, and VII. From this special case the general theorem may be deduced as follows:

By means of the special case of the theorem just referred to it is possible to find two values of $\lambda$ ($\lambda'$ and $\lambda''$) for each of which a solution of the differential equation exists which satisfies condition (1) and which vanishes when $x = b$ while the solution corresponding to $\lambda'$ has just $n - 1$ roots and the solution corresponding to $\lambda''$ just $n$ roots greater than $a$ and less than $b$. From Theorem I it is clear that $\lambda' > \lambda''$ and that when $\lambda' > \lambda \geqq \lambda''$ and only then the solution $y(x, \lambda)$ which satisfies condition (1) has just $n$ roots greater than $a$ and less than $b$. If then we can prove that in this interval there is one and only one value of $\lambda$ for which condition (2) is satisfied our theorem will be proved.

Now when $\lambda' > \lambda > \lambda''$ $y(b, \lambda) \gtrless 0$ as otherwise by Theorem

* See the preceding footnote.

I $y(x, \lambda'')$ would have at least $n+1$ roots greater than $a$ and less than $b$. Accordingly $y'(b, \lambda) / y(b, \lambda)$ is a continuous function of $\lambda$ between these limits. Moreover by Theorem II this ratio continually increases as $\lambda$ varies from $\lambda''$ to $\lambda'$, and it must increase from $-\infty$ to $+\infty$ since its numerical value clearly becomes infinite as $\lambda$ approaches either $\lambda'$ or $\lambda''$. The above mentioned ratio must therefore take on every value, and in particular the value $\beta'/\beta$, once and only once in the interval in question.

The case $n=0$ is not covered by the above proof. Here however we may as before find a value $\lambda''$ such that $y(b,\lambda'') = 0$ while $y(x, \lambda'')$ does not vanish when $a < x < b$, and it is clear as before that when $L > \lambda \geqq \lambda''$ and only then $y(x, \lambda) \neq 0$ when $a < x < b$, and that $y(b, \lambda) \neq 0$ when $L > \lambda > \lambda''$. Accordingly $y'(b, \lambda) / y(b, \lambda)$ is continuous and constantly increases as $\lambda$ varies from $\lambda''$ to $L$. Moreover its numerical value becomes infinite when $\lambda$ approaches $\lambda''$ and by the formula in Theorem IV the same is true when $\lambda$ approaches $L$, so that as before it takes on the value $\beta'/\beta$ for one and only one value of $\lambda$.

A special case of the last theorem deserves mention as it is the one which is of by far the greatest use, viz., the case in which $\alpha = 0$ or $\alpha' = 0$ and $\beta = 0$ or $\beta' = 0$. Here, as is at once seen by reference to the corollary stated after Theorem IV, we may replace the condition stated in VIII concerning $M$ by the condition that it must be possible to take $\lambda$ so near to $L$ that for all values of $x$ between $a$ and $b$ $\varphi(x, \lambda) \geqq 0$.

We added finally the theorem (whose truth is immediately obvious from Theorems I and II):

IX. *If two values ($\lambda_1$ and $\lambda_2$) of $\lambda$ are determined by Theorem VIII by using in each case the same values of $\alpha$ and $\alpha'$ and if the values of $n$, $\beta$, $\beta'$ corresponding to $\lambda_1$ are $n_1$, $\beta_1$, $\beta'_1$ and the values corresponding to $\lambda_2$ are $n_2$, $\beta_2$, $\beta_2'$ then:*

(1) *if $n_1 > n_2$ we shall have $\lambda_1 < \lambda_2$;*

(2) *if $n_1 = n_2$ and $\beta_1 = 0$ while $\beta_2 \neq 0$ then $\lambda_1 < \lambda_2$;*

(3) *if $n_1 = n_2$, $\beta_1 \neq 0$, $\beta_2 \neq 0$ and $\beta_1' / \beta_1 < \beta_2' / \beta_2$ then $\lambda_1 < \lambda_2$.**

We have assumed throughout this and the preceding section that $a < b$. The contrary assumption might have been treated in the same way although many of the theorems

*It is upon this theorem that Klein's determination of the relative magnitude of the $2n+1$ values of $B$ which in the theory of Lamé's polynomials correspond to one and the same value of $A = n(n+1)$ rests. *Cf.* Lineare Differentialgleichungen der zweiten Ordnung, p. 341–346. I take this occasion of mentioning the fact, which I pointed out to Professor Klein in June, 1893, that the theorem just referred to follows at once from the discussion of Poincaré, *Acta Mathematica*, vol. 7, p. 306.

would then require slight modification. The easiest way of getting the theorems in this case is perhaps to change the independent variable $x$ by the formula $x' = -x$.

## § 3. *Lamé's Equation.*

We will write Lamé's equation in the form :

$$\frac{d^2y}{dx^2} + \tfrac{1}{2}\left(\frac{1}{x-e_1} + \frac{1}{x-e_2} + \frac{1}{x-e_3}\right)\frac{dy}{dx} - \frac{Ax+B}{4(x-e_1)(x-e_2)(x-e_3)}y = 0.*$$

We shall regard the singular points $e_1$, $e_2$, $e_3$ as real and unequal ($e_1 < e_2 < e_3$) and constant, while $A$ and $B$ will be regarded as real parameters. The equation can be reduced to the binomial form :

$$\frac{d^2y}{dt^2} = (Ax+B)\,y$$

by the change of independent variable (see (G)) :

$$t = \int_c^x \frac{dx}{2\sqrt{(x-e_1)(x-e_2)(x-e_3)}}.$$

The lower limit of integration $c$ is arbitrary. Let us first consider a finite segment $ab$ of the $x$-axis where either $e_1 \leqq a < b \leqq e_2$ or $e_2 \leqq a < b$.† If we let in the first case $e_1 \leqq c \leqq e_2$ and in the second case $e_3 \leqq c$ the integral $t$ will be finite and real for all points of the segment $ab$. The segment $ab$ corresponds then in a one to one manner to the segment of the $t$-axis from $t(a)$ to $t(b)$. If then we notice that when we assign to $A$ an arbitrary value the function $Ax + B$ increases with $B$ for all points of the segment $t(a)$ to $t(b)$ and (to put it roughly) increases from $-\infty$ to $+\infty$ as $B$ varies from $-\infty$ to $+\infty$ we get at once from Theorem VIII :

X. *If we assign to $A$ an arbitrary value there will be one and only one value of $B$ ($B = F(A)$) for which Lamé's equation*

* For the relation between the notation here used and Lamé's notation see Reihenentwickelungen, p. 113, footnote.

† The following discussion applies equally well to the case in which the segment $ab$ covers a part or the whole of the interval $e_1\,e_2$ or $e_3\,\infty$ in which it lies more than once (provided merely that it remains of finite length), *cf.* Reihenentwickelungen, p. 123 and following. Only slight verbal alterations would be necessary in the text to include this case.

*has a solution $y$ which has $n$ roots in the segment $a < x < b$ and at the ends of the segment satisfies the relations:*

$$(1) \qquad \alpha' y(a) - \alpha \frac{dy(a)}{dt} = 0$$

$$(2) \qquad \beta' y(a) - \beta \frac{dy(a)}{dt} = 0.*$$

Here $n$ may be any integer positive or zero while $\alpha$, $\alpha'$, $\beta$, $\beta'$ are any real quantities provided that we have neither $\alpha = \alpha' = 0$ nor $\beta = \beta' = 0$. The function $F(A)$ depends, of course, upon all of these quantities, and if we indicate the argument $A$ only it is because in the questions which we shall take up the other quantities are regarded as fixed.† We proceed now to the deduction of some properties of the function $F(A)$ concerning which we, as yet, know only that it is defined and single valued for all values of $A$.

XI. *$A_1$ and $A_2$ being any two different values of $A$:*

$$a < \frac{F(A_2) - F(A_1)}{A_1 - A_2} < b.‡$$

For if we call the middle term of this inequality $x_0$ we have:

$$[A_1 x + F(A_1)] - [A_2 x + F(A_2)] = (A_1 - A_2)(x - x_0)$$

and if the above inequality did not hold we should have:

either $\quad A_1 x + F(A_1) \geqq A_2 x + F(A_2) \quad (a \leqq x \leqq b)$

or $\quad A_1 x + F(A_1) \leqq A_2 x + F(A_2) \quad (a \leqq x \leqq b)$

the equality sign holding at most when $x = a$ or when $x = b$. Either of these inequalities is seen to be impossible when we compare the equations:

$$\frac{d^2y}{dt^2} = [A_1 x + F(A_1)]y \qquad \frac{d^2y}{dt^2} = [A_2 x + F(A_2)]y$$

by means of Theorems I and II.

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* Except when $a$ or $b$ coincides with $e_1$, $e_2$, $e_3$ we may clearly replace $dy/dt$ in these conditions by $dy/dx$.

† A question in which this is not the case is the one referred to in the footnote on p. 306.

‡ This is nothing but an analytical statement of the fact that the two lines $y = A_1 x + F(A_1)$ and $y = A_2 x + F(A_2)$ intersect within the infinite strip bounded by the ordinates erected at $a$ and $b$. *Cf.* Reihenentwickelungen, p. 128, 1st line. The proof we here give is also nothing but an analytical statement of the proof there suggested.

XII. *$F(A)$ is continuous for every value of $A$.*

Let $A_1$ be the value of $A$ for which we wish to prove $F(A)$ continuous. Then we must prove that no matter how small the positive quantity $\delta$ may be chosen we can find a positive quantity $\varepsilon$ such that when

$$|A_2 - A_1| < \varepsilon \quad |F(A_2) - F(A_1)| < \delta.$$

Now consider the quantities $|a|$ and $|b|$ and call the largest of the two $h$. Then the last theorem shows that:

$$\frac{|F(A_2) - F(A_1)|}{|A_1 - A_2|} < h$$

and therefore if we let $\varepsilon = \delta / h$ the inequality we wish to establish is satisfied.*

We proceed now to four lemmas.

LEMMA 1. *It is possible to find a positive quantity $M$ so large that when $A > M$ $Ab + F(A) > 0$.*

For instance we may let

$$M = \frac{2(n+1)^2\pi^2}{(b-a)\left[t\left(\frac{a+b}{2}\right) - t(a)\right]^2}$$

For if there were a value of $A$ greater than this value for which $Ab + F(A) \leqq 0$ we should have for this value of $A$:

$$\frac{-(n+1)^2\pi^2}{\left[t\left(\frac{a+b}{2}\right) - t(a)\right]^2} > A\left(\frac{a-b}{2}\right) \geqq A\left(\frac{a+b}{2}\right) + F(A);$$

and therefore when $\quad a \leqq x \leqq \frac{a+b}{2}$

*i. e.*, when $\quad t(a) \leqq t \leqq t\left(\frac{a+b}{2}\right),$

$$Ax + F(A) < -\frac{(n+1)^2\pi^2}{\left[t\left(\frac{a+b}{2}\right) - t(a)\right]^2}.$$

*We have here proved that $F(A)$ is *uniformly* continuous (since the value found for $\varepsilon$ is independent of $A$) for all values of $A$. This would not follow from the mere continuity, since the interval in question is infinite.

Every solution of the equation:

$$\frac{d^2y}{dt^2} = [Ax + F(A)]\,y$$

must therefore by Theorem V have at least $n+1$ roots in the segment $t(a) < t < t(b)$ while by hypothesis one solution of this equation has only $n$ roots in this segment.

LEMMA 2. *It is possible to find a positive quantity $M$ so large that when $A < -M$ $Aa + F(A) > 0$.*

Letting

$$M = \frac{2(n+1)^2\pi^2}{(b-a)\left[t(b) - t\left(\frac{a+b}{2}\right)\right]^2}$$

the proof is precisely similar to the proof just given.

Up to this point we have placed no restriction on the constants $a$, $a'$, $\beta$, $\beta'$ except that neither $a$ and $a'$ nor $\beta$ and $\beta'$ shall both be zero. Now, however, in order to avoid all complication in the proofs and also because the cases to which we thus restrict ourselves are by far the most important *we will assume that either $a = 0$ or $a' = 0$ and either $\beta = 0$ or $\beta' = 0$, i. e.*, we assume $aa' = 0$ and $\beta\beta' = 0$. It is not hard to show that this restriction is not necessary for the truth of Theorem XIII (and therefore for the truth of Klein's theorem of oscillation) but the following two Lemmas would have to be slightly changed in form* and would be decidedly less easy to prove without this restriction.

LEMMA 3. *If $aa' = 0$ and $\beta\beta' = 0$ then when $A > 0$*

$$Aa + F(A) < 0.$$

For otherwise we should have when $a < x \leqq b$

$$Ax + F(A) > 0$$

and this is seen to be impossible from the Corollary to Theorem IV on account of the conditions at $a$ and $b$.

LEMMA 4. *If $aa' = 0$ and $\beta\beta' = 0$ then when $A < 0$*

$$Ab + F(A) < 0.$$

* The inequalities there given would then be true only when $A > M$ and $A < -M$ respectively when $M$ is a certain positive quantity. They are, however, true as stated, even without the restrictions on $a$, $a'$, $\beta$, $\beta'$ when $n \geqq 2$, and here the proof given in the text applies.

For otherwise we should have when $a \leqq x < b$

$$Ax + F(A) > 0$$

and this is seen to be impossible as before.

XIII. *If $\alpha\alpha' = 0$ and $\beta\beta' = 0$* it is possible to find a positive quantity $M$ such that when $A > M$ and also when $A < -M$:*

$$a < -\frac{F(A)}{A} < b.$$

This follows at once from the preceding Lemmas.

We have so far supposed the segment $ab$ to lie either in the interval from $e_1$ to $e_2$ or in the interval from $e_3$ to $+\infty$. Consider now a segment $ab$ where $a$ and $b$ satisfy one of the relations: $a < b \leqq e_1$ or $e_2 \leqq a < b \leqq e_3$. If then we take $c$ so that in the first case $c \leqq e_1$ in the second case $e_2 \leqq c \leqq e_3$ the integral $t$ will be pure imaginary throughout the segment $ab$. If we bring in the real variable $\tau$ by the relation $t = i\tau$ we get as the binomial form of the equation:

$$\frac{d^2y}{d\tau^2} = -(Ax + B)y.$$

The fact that $Ax + B$ has the negative instead of the positive sign will make some changes in the proofs of the foregoing propositions and in particular the statements of the four Lemmas will have to be somewhat changed, *but it is easy to see that Theorems X, XI, XII, XIII remain true if $\tau$ is substituted for $t$ in them.*

We are now in a position to prove Klein's *Theorem of Oscillation* (in its simplest form):

XIV. *Let $a_1 b_1$ and $a_2 b_2$ be any two segments of the $x$-axis such that $a_1 < b_1 \leqq a_2 < b_2$ and such that none of the inequalities $a_1 < e_i < b_1$ $(i = 1, 2, 3)$ or $a_2 < e_i < b_2$ $(i = 1, 2, 3)$ are true, and let us assign to each of the points $a_1$, $b_1$, $a_2$, $b_2$ one of the conditions $y = 0$ or $\frac{dy}{dt} = 0$; and let $n_1$ and $n_2$ be any two integers positive or zero. Then there exists one and only one pair of values $A$, $B$ for which Lamé's equation has a first solution which satisfies the prescribed conditions at $a_1$ and $b_1$ and has $n_1$ roots in the segment $a_1 < x < b_1$, and a second solution which satisfies the prescribed conditions at $a_2$ and $b_2$ and has $n_2$ roots in the segment $a_2 < x < b_2$.*

* These restrictions, as above stated, are not essential to the truth, but merely to the proof here given of the theorem.

Let us call the values of $B$ determined by applying Theorem X to the segments $a_1 b_1$ and $a_2 b_2$ respectively $F_1(A)$ and $F_2(A)$. Our theorem will be established if we can prove that there is one and only one value of $A$ for which $F_1(A) = F_2(A)$.

That there cannot be more than one such value is obvious at once from Theorem XI. For if $A_1$ and $A_2$ are two such values we find by applying Theorem XI in succession to the segments $a_1 b_1$ and $a_2 b_2$:

$$a_1 < \frac{F_1(A_2) - F_1(A_1)}{A_1 - A_2} < b_1 \leqq a_2 < \frac{F_2(A_2) - F_2(A_1)}{A_1 - A_2} < b_2.$$

But these inequalities contain a contradiction, since the second and fifth terms are equal.*

In order to show that there does exist one value of $A$ for which $F_1(A) = F_2(A)$ let us apply Theorem XIII in succession to the segment $a_1 b_1$ and $a_2 b_2$. We thus see that when $A$ has a sufficiently large positive value:

$$a_1 < -\frac{F_1(A)}{A} < b_1 \leqq a_2 < -\frac{F_2(A)}{A} < b_2,$$

and therefore: $F_1(A) > F_2(A)$. On the other hand, when $A$ is negative and its numerical value is sufficiently large we see in the same way that: $F_1(A) < F_2(A)$. Accordingly $F_1(A) - F_2(A)$ is positive when $A$ has a large positive value and negative when $A$ has a numerically large negative value and must, therefore, since by Theorem XII it is continuous for all values of $A$, be zero for some value of $A$.†

As has already been mentioned the theorem of oscillation still holds if for the simple conditions at $a_1$, $b_1$, $a_2$, $b_2$ we substitute more complicated ones of the form $a'y - a\, dy/dt = 0$, and in fact the proof of the negative part of the theorem applies immediately to this case as does also the proof of the positive part when $n_1 \geqq 2$ and $n_2 \geqq 2$; and even in the special cases in which $n_1$ or $n_2$ has the value 0 or 1 the proof requires no change when once Theorem XIII has been

*For a geometrical statement of this proof see Reihenentwickelungen, p. 130, second footnote.

† Geometrically this proof means that since the line $y = Ax + B$ will, when sufficiently steep, cut the axis of $x$ in the segment $a_1 b_1$ or $a_2 b_2$ according as one or the other of these segments is used to determine it, the line determined by $a_1 b_1$ will when $A$ is large and positive liea bove, when $A$ is numerically large and negative lie below, the line determined by $a_2 b_2$ and must therefore for some intermediate value of $A$ coincide with it.

established for these cases. On the other hand the theorem of oscillation for other differential equations which like Lamé's involve two parameters* may be established by reasoning almost identical with that here used, the difference again coming in only in the four Lemmas. I hope soon to return to these and other similar questions.

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# SOME EXAMPLES OF DIFFERENTIAL INVARIANTS.

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(Read before the American Mathematical Society at the Meeting of December 29, 1897.)

IN the following paper certain invariants for projective transformations are given. The derivation, according to Lie's methods, is given in full for the plane, and the method for the corresponding problem in space of three dimensions is sketched in, and the results of the solution are given. It is believed that all the invariants given are new.

For an infinitesimal point transformation of the $xy$ plane $x$ and $y$ receive the increments

$$\delta x = \xi(x, y)\,\delta t, \quad \delta y = \eta(x, y)\,\delta t,$$

respectively, where $\delta t$ is an infinitesimal independent of $x$ and $y$. This infinitesimal transformation is represented by the symbol

$$Xf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}.$$

The increment of any function $\varphi(x, y)$ is then

$$\delta\varphi = \frac{\partial \varphi}{\partial x}\delta x + \frac{\partial \varphi}{\partial y}\delta y = \left(\frac{\partial \varphi}{\partial x}\xi + \frac{\partial \varphi}{\partial y}\eta\right)\delta t = X\varphi \cdot \delta t.$$

If, then, $\varphi$ is to be invariant for the transformation $Xf$, we have as a necessary and sufficient condition $X\varphi = 0$. Lie

*For instance Lamé's generalized equation. See Reihenentwickelungen, p. 125.

has further shown that if $\varphi(x, y)$ be invariant for the infinitesimal transformation $Xf$, that it is also invariant for the finite transformations defined by $Xf$. This follows at once from the fact that any one of these finite transformations changes $\varphi(x, y)$ into $\varphi_1$, where*

$$\varphi_1 = \varphi + \frac{t}{1!} X\varphi + \frac{t^2}{2!} X(X\varphi) + \frac{t^3}{3!} X[X(X\varphi)] + \cdots$$

If $X\varphi = 0$ this reduces to $\varphi_1 = \varphi$, so that $\varphi(x, y)$ is invariant for the finite transformation. When, therefore, we desire to find the invariants for the infinitesimal transformation $Xf$ we have simply to solve the partial differential equation

$$\xi(x, y) \frac{\partial \varphi}{\partial x} + \eta(x, y) \frac{\partial \varphi}{\partial y} = 0.$$

In this case there is only one invariant, all others being functions of this one.

The transformation $Xf$ transforms not only the points $(x, y)$, but also every figure in the plane. Thus if we consider a curve, its points, slope, radius of curvature, etc., are transformed. The manner in which the points are transformed is given by $Xf$, and knowing how the individual points are transformed we can compute how

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \cdots$$

are transformed. It is easily shown that†

$$\delta y' = \left[\frac{d\eta}{dx} - y' \frac{d\xi}{dx}\right] \delta t$$

$$= \left[\frac{\partial \eta}{\partial x} + y' \left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right) - y'^2 \frac{\partial \xi}{\partial y}\right] \delta t = \eta_1(x, y, y') \delta t,$$

$$\delta y'' = \left[\frac{d\eta_1}{dx} - y'' \frac{d\xi}{dx}\right] \delta t = \eta_2(x, y, y', y'') \delta t.$$

The symbol

$$X'f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta_1 \frac{\partial f}{\partial y}$$

* Lie: Differentialgleichungen mit bekannten infinitesimalen Transformationen, herausgegeben von Scheffers, p. 58, Theorem 4, and p. 62.

† Lie: Differentialgleichungen, p. 272, p. 358.

shows then how $x$, $y$, and $y'$ are transformed, and therefore how any function of them is transformed. This is known as the *once extended* transformation. Just as before the increment of $\varphi(x, y, y')$ is $X'\varphi \cdot \delta t$. Similarly, the *twice* extended transformation is

$$X''f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta_1 \frac{\partial f}{\partial y'} + \eta_{12} \frac{\partial f}{\partial y''},$$

and $X''\varphi \cdot \delta t$ is the increment of $\varphi(x, y, y', y'')$. To find the *differential invariants* of the $n$th order for the transformation $Xf$ we must extend $Xf$ $n$ times, and then the solutions of the partial differential equation $X^{(n)}f = 0$ are the desired invariants. In solving this equation $x, y, y', y'', \cdots, y^{(n)}$ are to be treated as independent variables. If the function is to be a simultaneous invariant for a number of transformations $X_1f, X_2f, \cdots, X_rf$, we have to seek the solutions of the simultaneous system of partial differential equations

$$X_1^{(n)}f = 0, \; X_2^{(n)}f = 0, \cdots, X_r^{(n)}f = 0.$$

Of course there may be no common solutions of this system of differential equations. In particular, if $X_1f, \cdots, X_rf$ are the infinitesimal transformations of a group with $r$ parameters, the system of differential equations just mentioned is complete. The number of variables involved is $n + 2$, and therefore in this case there are at least $n + 2 - r$ common solutions of the $r$ equations. If the equations are linearly dependent there are more solutions, there being $n + 2 - r + k$, when the equations are connected by $k$ linear relations.

If we consider a configuration involving a number of points $(x_1, y_1)$, $(x_2, y_2)$, $\cdots$, instead of only one point, we must write instead of $Xf$ the following:

$$\xi(x_1 y_1) \frac{\partial f}{\partial x_1} + \eta(x_1 y_1) \frac{\partial f}{\partial y_1} + \xi(x_2 y_2) \frac{\partial f}{\partial x_2} + \eta(x_2 y_2) \frac{\partial f}{\partial y_2} + \cdots$$

To find the differential invariants $\varphi(x_1, y_1, x_2, y_2, y_1', y_2', y_1'', y_2'', \cdots)$ this transformation must be extended just as before, and the solutions sought of the partial differential equation obtained by equating this extended transformation to zero.

We have now stated the principles required for the solution of the following problem:

*Given two curves in the xy plane, with a point on each. The plane is subjected to the general projective transformation. Required the differential invariants of the second order.*

It is well known that the general projective group in the plane has eight parameters, and that the eight independent infinitesimal transformations may be written in the form*

$$X_1 f = \frac{\partial f}{\partial x}, \quad X_2 f = \frac{\partial f}{\partial y}, \quad X_3 f = x\frac{\partial f}{\partial x}, \quad X_4 f = x\frac{\partial f}{\partial y}, \quad X_5 f = y\frac{\partial f}{\partial x},$$

$$X_6 f = y\frac{\partial f}{\partial y}, \quad X_7 f = x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y}, \quad X_8 f = xy\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y}.$$

When we consider the differential invariants of the second order for two points, we have eight variables, viz.

$$x_1,\ y_1,\ y_1',\ y_1'',\ x_2,\ y_2,\ y_2',\ y_2''.$$

Therefore by extending each of the transformations

$$X_1 f, \cdots, X_8 f$$

twice we obtain a complete system of eight equations involving eight variables. The transformations are extended by means of the formulæ

$$\delta y' = \left[\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right]\delta t = \eta_1 \delta t,$$

$$\delta y'' = \left[\frac{d\eta_1}{dx} - y''\frac{d\xi}{dx}\right]\delta t = \eta_2 \delta t.$$

For example, for $X_8 f$ we have, $\xi = xy$, $\eta = y^2$, and

$$\eta_1 = 2yy' - y'(xy' + y) = yy' - xy'^2, \quad \eta_2 = -3xy'y''.$$

The complete system of partial differential equations is thus found to be:

$$0 = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2},$$

$$0 = \frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial y_2},$$

$$0 = x_1\frac{\partial f}{\partial x_1} - y_1'\frac{\partial f}{\partial y_1'} - 2y_1''\frac{\partial f}{\partial y_1''} + x_2\frac{\partial f}{\partial x_2} - y_2'\frac{\partial f}{\partial y_2'} - 2y_2''\frac{\partial f}{\partial y_2''},$$

$$0 = x_1\frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial y_1'} + x_2\frac{\partial f}{\partial y_2} + \frac{\partial f}{\partial y_2''},$$

* Lie: Continuierliche Gruppen, herausgegeben von Scheffers, p. 26.

$$0 = y_1 \frac{\partial f}{\partial x_1} - y_1'^2 \frac{\partial f}{\partial y_1'} - 3\, y_1' y_1'' \frac{\partial f}{\partial y_1''}$$

$$+ y_2 \frac{\partial f}{\partial x_2} - y_2'^2 \frac{\partial f}{\partial y_2'} - 3\, y_2' y_2'' \frac{\partial f}{\partial y_2''},$$

$$0 = y_1 \frac{\partial f}{\partial y_1} + y_1' \frac{\partial f}{\partial y_1'} + y_1'' \frac{\partial f}{\partial y_1''} + y_2 \frac{\partial f}{\partial y_2} + y_2' \frac{\partial f}{\partial y_2'} + y_2'' \frac{\partial f}{\partial y_2''},$$

$$0 = x_1^2 \frac{\partial f}{\partial x_1} + x_1 y_1 \frac{\partial f}{\partial y_1} + (y_1 - x_1 y_1') \frac{\partial f}{\partial y_1'} - 3\, x_1 y_1'' \frac{\partial f}{\partial y_1''}$$

$$+ x_2^2 \frac{\partial f}{\partial x_2} + x_2 y_2 \frac{\partial f}{\partial y_2} + (y_2 - x_2 y_2') \frac{\partial f}{\partial y_2'} - 3\, x_2 y_2'' \frac{\partial f}{\partial y_2''},$$

$$0 = x_1 y_1 \frac{\partial f}{\partial x_1} + y_1^2 \frac{\partial f}{\partial y_1} + y_1' (y_1 - x_1 y_1') \frac{\partial f}{\partial y_1'} - 3\, x_1 y_1' y_1'' \frac{\partial f}{\partial y_1''}$$

$$+ x_2 y_2 \frac{\partial f}{\partial x_2} + y_2^2 \frac{\partial f}{\partial y_2} + y_2' (y_2 - x_2 y_2') \frac{\partial f}{\partial y_2'} - 3\, x_2 y_2' y_2'' \frac{\partial f}{\partial y_2''}.$$

If these eight equations are independent there are no common solutions, and consequently no invariant of the second order. It is found however, that the determinant of the coefficients vanishes identically, that is:

$$\begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
x_1 & 0 & -y_1' & -2y_1'' & x_2 & 0 & -y_2' & -2y_2'' \\
0 & x_1 & 1 & 0 & 0 & x_2 & 1 & 0 \\
y_1 & 0 & -y_1'^2 & -3y_1'y_1'' & y_2 & 0 & -y_2'^2 & -3y_2'y_2'' \\
0 & y_1 & y_1' & y_1'' & 0 & y_2 & y_2' & y_2'' \\
x_1^2 & x_1y_1 & y_1 - x_1y_1' & -3x_1y_1'' & x_2^2 & x_2y_2 & y_2 - x_2y_2' & -3x_2y_2'' \\
x_1y_1 & y_1^2 & y_1'(y_1 - x_1y_1') & -3x_1y_1'y_1'' & x_2y_2 & y_2^2 & y_2'(y_2 - x_2y_2') & -3x_2y_2'y_2''
\end{vmatrix} \equiv 0$$

Therefore the equations are connected by a linear relation. Moreover, all the seven rowed determinants do not vanish identically; for instance, the upper left hand corner one is equal to

$$y_1'' [y_2 - y_1 - y_1' (x_2 - x_1)] [y_2 - y_1 - y_2' (x_2 - x_1)]$$

$$[y_2 - y_1 - y_1' (x_2 - x_1) + y_2 - y_1 - y_2' (x_2 - x_1)].$$

Seven of the equations are therefore linearly independent, and consequently there is one and only one common solution of this system of partial differential equations. By

the method of finding all the solutions of one of the equations, and substituting them as new independent variables in the remaining equations, and repeating the process, this one solution is readily shown to be

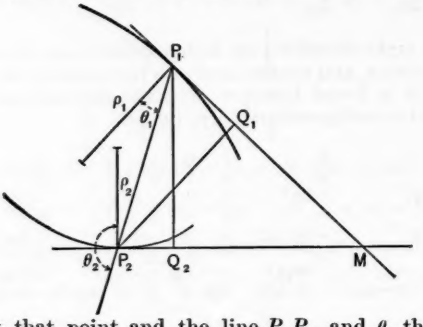
$$f = \frac{y_1''}{y_2''}\left[\frac{y_2 - y_1 - y_2'(x_2 - x_1)}{y_2 - y_1 - y_1'(x_2 - x_1)}\right]^3$$

This is the only invariant of the second order of two points, for the general projective transformation of the plane. To interpret this geometrically, we have:

$$\frac{[y_2 - y_1 - y_2'(x_2 - x_1)]^3}{y_2''} = \frac{(1 + y_2')^{\frac{3}{2}}}{y_2''}\left[\frac{y_2 - y_1 - y_2'(x_2 - x_2)}{(1 + y_2')^{\frac{1}{2}}}\right]^3$$

$$= \rho_1 \cdot \overline{P_1 Q_1}^3,$$

where $\rho_2$ is the radius of curvature at $(x_2, y_2)$, or $P_2$, of the curve through this point, and $\overline{P_1 Q_2}$ is the length of the perpendicular let fall from $(x_1, y_1)$, or $P_1$, to the tangent at $(x_2, y_2)$. Let the tangents at $P_1$ and $P_2$ intersect at $M$ (see figure), $\theta_1$ be the angle at $P_1$ between the normal to the




curve at that point and the line $P_1 P_2$, and $\theta_2$ the corresponding angle at $P_2$. Then the invariant is

$$\frac{\rho_2}{\rho_1}\left(\frac{P_1 Q_2}{P_2 Q_1}\right)^3 = \frac{\rho_2}{\rho_1}\left(\frac{P_1 M}{P_2 M}\right)^3 = \frac{\rho_2 \cos^3 \theta_2}{\rho_1 \cos^3 \theta_1}.$$

*Discussion of the Invariant for a Special Case.*

Let us suppose that the two points lie on the same curve, and that the invariant has a constant value however the points be chosen. Interchanging the two points the invariant takes on the reciprocal value, hence its value must be $\pm 1$. Now let the points be taken infinitely near to each

other. Then, dropping the subscripts for the point $(x_1, y_1)$, we have

$$x_2 = x + \delta x,$$

$$y_2 = y + y'\delta x + \frac{y''}{2!}\delta x^2 + \frac{y'''}{3!}\delta x^3 + \frac{y^{iv}}{4!}\delta x^4 + \frac{y^v}{5!}\delta x^5 + \cdots,$$

$$y_2' = y' + y''\delta x + \frac{y'''}{2!}\delta x^2 + \frac{y^{iv}}{3!}\delta x^3 + \frac{y^v}{4!}\delta x^4 + \cdots,$$

$$y_2'' = y'' + y'''\delta x + \frac{y^{iv}}{2!}\delta x^2 + \frac{y^v}{3!}\delta x^3 + \cdots,$$

and

$$\frac{y_2''}{y''}\left[\frac{y_2 - y - y'(x_2 - x)}{y_2 - y - y_2'(x_2 - x)}\right]^3$$

$$= -1 - \frac{\delta x^3}{y''^3}\left[9y^v y'' - 45y''y'''y^{iv} + 40y'''^3\right] + \cdots$$

This shows that the invariant cannot have the constant value $+1$; if it has the constant value $-1$ the two points must lie on the curve whose differential equation is

$$9y^v y'' - 45y''y'''y^{iv} + 40y'''^3 = 0,$$

that is, they must lie on a conic. Conversely, if the two points do lie on a conic the value of the invariant is easily shown to be $-1$. The case in which the conic degenerates into a pair of right lines has no interest, as the invariant then becomes indeterminate. We have, therefore:

*If two points lie on a curve and the invariant*

$$\frac{\rho_2 \cos^3 \theta_2}{\rho_1 \cos^3 \theta_1}$$

*has a constant value for all positions of the points on the curve, then this constant value must be $-1$, and the curve is an undegenerate conic.*

That is, we have for a conic

$$\rho_1 \cos^3 \theta_1 + \rho_2 \cos^3 \theta_2 = 0.$$

This latter result may also be deduced as a special case of a general theorem due to Reiss.* Let an algebraic curve of the $n$th degree be cut by any right line in the points $M_1, M_2, \cdots, M_n$. Denote the radius of curvature at the point $M_i$ by $\rho_i$, and the angle between the normal at $M_i$ and the transversal by $\theta_i$. Then the formula referred to is

$$\sum_1^n \frac{1}{\rho_i \cos^3 \theta_i} = 0.$$

*Correspondance mathématique et physique de Quetelet, vol IX., p. 152.

When the curve is a conic this evidently reduces to

$$\rho_1 \cos^3 \theta_1 + \rho_2 \cos^3 \theta_2 = 0.$$

That, in general, $\dfrac{\rho_2 \cos^3 \theta_2}{\rho_1 \cos^3 \theta_1}$ is invariant for all projective transformations of the plane, whether the two points be on the same curve or different curves, seems to have been hitherto unnoticed.

*Analogous Invariant for Surfaces.*

In space of three dimensions consider two surfaces, and a point on each. Let the surfaces be subjected to the general projective transformation, and let us seek the differential invariants of the second order. The general projective group in three dimensions has fifteen infinitesimal transformations. If we write

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2},$$

the invariant will be a function of $x_1, y_1, z_1, p_1, q_1, r_1, s_1, t_1$, and $x_2, \cdots, t_2$; that is, it will involve sixteen variables. Each of the fifteen infinitesimal transformations must then be twice extended. If the point transformation be

$$Xf = \xi(x, y, z) \frac{\partial f}{\partial x} + \eta(x, y, z) \frac{\partial f}{\partial y} + \zeta(x, y, z) \frac{\partial f}{\partial x},$$

the twice extended transformation is

$$X''f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} + \varphi \frac{\partial f}{\partial p} + \psi \frac{\partial f}{\partial q} + \rho \frac{\partial f}{\partial r} + \sigma \frac{\partial f}{\partial s} + \tau \frac{\partial f}{\partial t},$$

where we have,

$$\varphi = \frac{\partial \zeta}{\partial x} + p \frac{\partial \zeta}{\partial z} - p\left(\frac{\partial \xi}{\partial x} + p \frac{\partial \xi}{\partial z}\right) - q\left(\frac{\partial \eta}{\partial x} + p \frac{\partial \eta}{\partial z}\right),$$

$$\psi = \frac{\partial \zeta}{\partial y} + q \frac{\partial \zeta}{\partial z} - p\left(\frac{\partial \xi}{\partial y} + q \frac{\partial \xi}{\partial z}\right) - q\left(\frac{\partial \eta}{\partial y} + q \frac{\partial \eta}{\partial z}\right),$$

$$\rho = \frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial z} + r \frac{\partial \varphi}{\partial p} + s \frac{\partial \varphi}{\partial q} - r\left(\frac{\partial \xi}{\partial x} + p \frac{\partial \xi}{\partial z}\right) - s\left(\frac{\partial \eta}{\partial x} + p \frac{\partial \eta}{\partial z}\right),$$

$$\sigma = \frac{\partial \varphi}{\partial y} + q\frac{\partial \varphi}{\partial z} + s\frac{\partial \varphi}{\partial p} + t\frac{\partial \varphi}{\partial q}$$
$$- r\left(\frac{\partial \xi}{\partial y} + q\frac{\partial \xi}{\partial z}\right) - s\left(\frac{\partial \eta}{\partial y} + q\frac{\partial \eta}{\partial z}\right),$$

or
$$\sigma = \frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z} + r\frac{\partial \psi}{\partial p} + s\frac{\partial \psi}{\partial q}$$
$$- s\left(\frac{\partial \xi}{\partial x} + p\frac{\partial \xi}{\partial z}\right) - t\left(\frac{\partial \eta}{\partial x} + p\frac{\partial \eta}{\partial z}\right),$$
$$\tau = \frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial z} + s\frac{\partial \psi}{\partial p} + t\frac{\partial \psi}{\partial q}$$
$$- s\left(\frac{\partial \xi}{\partial y} + q\frac{\partial \xi}{\partial z}\right) - t\left(\frac{\partial \eta}{\partial y} + q\frac{\partial \eta}{\partial z}\right).$$

Using these formulas to extend the group, we obtain a complete system of 15 partial differential equations in 16 variables. The equations are, however, *not independent*, there being one relation between them. They have therefore $16 - 15 + 1 = 2$ solutions, and therefore there are two invariants of the second order. One of these is at once deducible geometrically. At each of the points draw the two principal tangents to the surface. Pass a plane through the line joining the two points, and each of these tangents. Then, since principal tangents are carried over by a projective transformation into principal tangents, the anharmonic ratio of these four planes must be invariant. This first invariant, which we shall call $I_1$, has no great interest. The second invariant may also be derived geometrically, by making use of the invariant found for plane curves, but the process is very long. It is much shorter to actually integrate the system of fifteen differential equations above mentioned. We find that

$$I_2 = \frac{s_1^2 - r_1 t_1}{s_2^2 - r_2 t_2}\left[\frac{z_2 - z_1 - p_2(x_2 - x_1) - q_2(y_2 - y_1)}{z_2 - z_1 - p_1(x_2 - x_1) - q_1(y_2 - y_1)}\right]^4$$

is the second solution. The geometrical meaning of this is that

$$I_2 = \frac{R_1 R_1' \cos^4 \theta_1}{R_2 R_2' \cos^4 \theta_2}$$

is invariant for a projective transformation, where $R_1$, $R_1'$ are the principal radii of curvature at $(x_1, y_1, z_1)$, and $\theta_1$ is the angle between the normal at $(x_1, y_1, z_1)$ and the line joining $(x_1, y_1, z_1)$ with $(x_2, y_2, z_2)$; and $R_2$, $R_2'$ and $\theta_2$ are the corresponding quantities at $(x_2, y_2, z_2)$.

If the two points lie on one surface, and the invariant $I_2$ have a constant value however the points be chosen on this surface, then $I_2 = +1$. If the surface be a quadric surface which is not a cone, it is easily shown that $I_2 = 1$.

The analytical expression for the first invariant, $I_1$, is the following :

$$I_1 = \frac{\left\{\begin{array}{l} (z_2-z_1)^2(r_1t_2+r_2t_1-2s_1s_2) \\ +2(z_2-z_1)(x_2-x_1)[(p_1+p_2)s_1s_2-p_1r_2t_1-p_2r_1t_2 \\ \qquad +(q_2-q_1)(r_1s_2-r_2s_1)] \\ +2(z_2-z_1)(y_2-y_1)[(q_1+q_2)s_1s_2-q_1r_1t_2-q_2r_2t_1 \\ +(p_2-p_1)(s_2t_1-s_1t_2)]+(x_2-x_1)^2[p_1^2r_2t_1+p_2^2r_1t_2 \\ -2p_1p_2s_1s_2+2(q_2-q_1)(p_1r_2s_1-p_2r_1s_2)+(q_2-q_1)^2r_2r_1] \\ +2(x_2-x_1)(y_2-y_1)\left[\begin{array}{l}(p_1q_2+p_2q_1-2p_1q_1-2p_2q_2)s_1s_2 \\ +p_1q_2r_2t_1+p_2q_1r_1t_2 \\ +(p_2-p_1)(p_2s_1t_2-p_1s_2t_1) \\ +(q_2-q_1)(q_2r_2s_1-q_1r_1s_2)\end{array}\right] \\ +(y_2-y_1)^2[q_1^2r_1t_2+q_2^2r_2t_1-2q_1q_2s_1s_2 \\ \qquad +2(p_2-p_1)(q_1s_1t_2-q_2s_2t_1)+(p_2-p_1)^2t_1t_2] \end{array}\right\}}{\begin{array}{l}[z_2-z_1-p_1(x_2-x_1)-q_1(y_2-y_1)]\sqrt{(s_1^2-r_1t_1)} \\ [z_2-z_1-p_2(x_2-x_1)-q_2(y_2-y_1)]\sqrt{(s_2^2-r_2t_2)}\end{array}}$$

This is *not* the anharmonic ratio of the four planes mentioned above, but that ratio is

$$-\tfrac{1}{4}I_1+\tfrac{1}{2}.$$

This anharmonic ratio is readily computed when we notice that the anharmonic ratio of four planes, all passing through two common points $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$, and one plane through each of the four points $(x_1, y_1, z_1), \cdots, (x_4, y_4, z_4)$ is

$$\frac{D_{12}D_{34}}{D_{14}D_{32}}$$

where

$$D_{ik} = \begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ x_i & y_i & z_i & 1 \\ x_k & y_k & z_k & 1 \end{vmatrix}$$

If the two points lie on a quadric (not a cone) we find $I_1 = 2$, so that the anharmonic ratio of the four planes is zero. The two pairs of planes reduce to a single pair for quadric surfaces, as is also evident from geometrical considerations.

LEIPZIG,
*December* 7, 1897.

# ON AN EXTENSION OF SYLOW'S THEOREM.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

SINCE we shall employ Cauchy's theorem * in what follows it seems desirable to give a simple proof of it. It may be stated as follows: *A group $G$ whose order $g$ is divisible by a prime number $p$ contains an operator of order $p$.*

We shall first suppose that $G$ is Abelian. If it is generated by a single operator $S$ of order $np$, we have $S^n \neq 1$ and $(S^n)^p = 1$. Hence $S^n$ is the required operator. If $G$ cannot be generated by a single operator we may represent a set of generating operators by $S_1, S_2, \cdots, S_r$. Since these generators are commutative the order of the group which any number of them generate cannot be divisible by any prime number that is not contained in the order of at least one of the generators. Hence the order of at least one of the given generators of $G$ must be divisible by $p$, and some power of this generator must be the required operator of order $p$.

We may now suppose that $G$ is a non-Abelian group of order $np$, and that our theorem is proved for all Abelian groups and for all non-Abelian groups whose orders are less than $np$. Let $g_1$ be the order of the largest subgroup of $G$ that transforms a given operator into itself; $g \div g_1$ is the number of conjugates of this operator. Hence

$$g = \frac{g}{g_1} + \frac{g}{g_2} + \cdots + \frac{g}{g_k} \qquad (A)$$

$k$ being the number of systems of conjugate operators and $g_1, g_2, \cdots, g_k$ being the orders of the largest subgroups that transform one operator of each system into itself. The operators for which $g_\alpha = g$ form an Abelian subgroup of $G$. If the order of this subgroup is not divisible by $p$ some $g_\beta < g$ must be divisible by $p$, since the second member of (A) must be divisible by this number. The main features of this method of proof are due to Frobenius.

THEOREM I. *If a group $G$ contains $r$ $(r > 0)$ subgroups $G_1, G_2, \cdots, G_r$ of order $p_1^\alpha p_2^\beta p_3^\gamma \cdots$ ($p_1, p_2, p_3, \cdots$ being different*

* Cauchy: Exercises d'analyse, III (1844), p. 250. Cf. Jordan: Traité des substitutions (1870), p. 26.

*prime numbers) and if G contains no subgroup whose order may be obtained by merely increasing the power of any of these prime numbers, then will each of these r subgroups transform all the others according to a substitution group S whose degree is exactly $r-1$ and which is isomorphic to the transforming group. Hence* $r = 1 + k_1 p_1 + k_2 p_2 + k_3 p_3 + \cdots$ *($k_1$, $k_2$, $\cdots$ being positive factors of* $p_1^{\alpha} p_2^{\beta} p_3^{\gamma} \cdots$*).*

*If the degree of each of the transitive constituents of every S is divisible by the same prime factor, say* $p_1$*, the r given subgroups are conjugate, and the order of G is divisible by* $p_1^{\alpha}$ *but is not divisible by* $p_1^{\alpha+1}$*.*

If all the operators of one of the given $r$ subgroups $G_1$ transformed another one of them into itself this second subgroup and an operator of $G_1$ that is not contained in the second would generate a group whose order could be obtained by merely increasing the power of one or more of the factors of the order of $G_1$. As this is contrary to the hypothesis each one of the given $r$ subgroups must transform all of the others according to some isomorphic substitution group of degree $r-1$. When $G$ is a simple group this isomorphism must be simple. The degree of each of the transitive constituents of $S$ must be a divisor of the order of this constituent and hence it must also be a divisor of the order of $S$. This proves the first part of the theorem.

When the degree of each of the transitive constituents of any one $S$ is divisible by $p_1$, $r \equiv 1 \bmod p_1$ according to the preceding paragraph. We proceed to prove that $G$ transforms the given $r$ subgroups according to a transitive substitution group $S_1$, whenever the transitive constituents of each of the $r$ $S$'s satisfy the given condition. If $S_1$ were intransitive, any one of the given $r$ subgroups that did not correspond to an element of a given transitive constituent of $S_1$ would transform the elements of this constituent in sets containing multiples of $p_1$ and such transforming subgroups could be found for each of the transitive constituents of $S_1$. This is impossible since $r$ is not divisible by $p_1$. Hence $S_1$ is transitive, *i. e.*, the given $r$ subgroups are conjugate when the degree of each of the transitive constituents of every $S$ is divisible by the same prime number.

We shall now consider the largest subgroup that transforms one of the given subgroups, say $G_1$, into itself. The order $q$ of its quotient group with respect to $G_1$ cannot be divisible by a factor of the order of $G_1$, otherwise there would be a subgroup whose order could be obtained by merely increasing the order of one or more of the prime factors of the order of $G_1$. The order of the largest sub-

group that transforms $G_1$ into itself must therefore be $q p_1^{\alpha} p_2^{\beta} p_3^{\gamma} \cdots$ and $g$ must be the product of this order and the number of conjugates of $G_1$; *i. e.*,

$$g = r q p_1^{\alpha} p_2^{\beta} p_3^{\gamma} \cdots$$

When the given condition is satisfied $rq$ is prime to $p_1$, hence $p_1^{\alpha}$ is the highest power of $p_1$ that divides $g$.

COROLLARY I. *When the order of a group is divisible by $p^{\beta}$ but not by $p^{\beta+1}$ the group contains $1 + kp$ subgroups of this order. All these subgroups are conjugate.*

In the preceding theorem we may let $p = p_1$, and $1 = p_2 = p_3 = \cdots$, for Cauchy's theorem assures the existence of a subgroup of order $p^{\alpha}$, $\alpha > 0$, when $\beta > 0$. Since $S$ is isomorphic to a group of order $p^{\alpha}$ the degree of each of its transitive constituents must be a power of $p$, *i. e.*, $r = 1 + kp$ and $\alpha = \beta$. This is known as Sylow's theorem. By restricting ourselves to this special case in the proof of the above theorem we obtain a simple method of proving the fundamental theorem of Sylow. The steps are as follows: (1) We observe from Cauchy's theorem that there is a subgroup of order $p^{\alpha}$, $\alpha > 0$. (2) If $p^{\alpha}$ is the highest order of such a subgroup the number of the subgroups of this order is $\equiv 1 \bmod p$. (3) All these subgroups are conjugate. (4) $p^{\alpha}$ is the highest power of $p$ that divides the order of the group, *i. e.*, $\alpha = \beta$.

It will be observed that the order of these steps is different from that adopted in the recent works on groups. It seems that this method of proof would be more desirable than the one generally given since it is not less simple and it gives due credit to Cauchy's important contribution to the theorem which bears the name of Sylow.

COROLLARY II. *If one of the given $r$ subgroups transforms all the others according to a simply isomorphic substitution group each of them has this property.*

For if an operator of one of these subgroups transforms all the others into themselves it must be contained in each of them.

The value of $r$ can clearly not exceed the quotient obtained by dividing the order of $G$ by that of $G_1$. This elementary condition, combined with the given form of $r$, is sometimes sufficient to restrict the value of $r$ to a small number of values, *e. g.*, if a group of order 105 contains a subgroup of order 21 the number ($r$) of these subgroups must satisfy the relations $r = 1 + 7k_1 + 3k_2$, $r < 6$. Hence $r = 1$ or 4.

As a simple illustration of the given theorem we may consider the subgroups of order $p_1 p_2$ that are contained in a substitution group of degree $p_1$, $p_1 > p_2$, that does not contain any subgroup of order $p_1 p_2^{\alpha}$, $\alpha > 1$. If any one of these subgroups, $G_1$, had a substitution of order $p_1$ in common with another, $G_2$, these two subgroups would generate a metacyclic group of order $p_1 p_2^{\alpha}$. As this is contrary to the hypothesis the number of these subgroups must $\equiv 1 \bmod p_1$ and all of them must be conjugate.

THEOREM II. *If any group $G$ of order $g$ contains a subgroup of order $p^{\alpha}q^{\beta}$, $p$ and $q$ being any prime numbers, the number of its subgroups of this order is of the form $1 + kp + lq$, $k$ and $l$ being positive integers.*

We may assume that $\alpha, \beta > 0$ since it is known that $k = 0$ when $\alpha = 0$* and that $k$ and $l = 0$ when $\alpha$ and $\beta = 0$. We may also assume that the order of $G$ is $p^{\alpha_1}q^{\beta_1}$, for if $g$ contained a prime factor besides $p$ and $q$, $G$ contained a subgroup of order $p^{a}q^{b}$ where $a$ and $b$ are both maxima. This subgroup would transform all the subgroups of order $p^{\alpha}q^{\beta}$ that are not included in it according to an isomorphic substitution group. Hence the number of the latter subgroups would be $a_1 p + bq$ ($a_1$ and $b_1$ being positive integers) and it would only be necessary to prove that the number of these subgroups in the group of order $p^{a}q^{b}$ is $1 + a_2 p + b_2 q$. In what follows we shall therefore assume $\alpha, \beta > 0$ and $g = p^{\alpha_1}q$ .

As the theorem is evidently true when $\alpha_1 = \beta_1 = 1$ we may assume that it is true with respect to every group of order $p^{\alpha'}q^{\beta'}$ when $p^{\alpha_1}q^{\beta_1} \div p^{\alpha'}q^{\beta'}$ is an integer which exceeds unity. The subgroups of order $p^{\alpha}q^{\beta}$ may be represented $G_1, G_2, \cdots, G_r$ and we may assume that these $r$ subgroups generate $G$. If some one of them transforms each one of the others into a different group the theorem is evidently true. In general, we multiply a sufficient number of them together so that the product transforms each one of the subgroups of the given order that is not contained in it. This product may be $G$ itself. At least one of the given subgroups $G_\alpha$ is self-conjugate in this product.

We now consider the number of the given subgroups which have more than one operator in common with $G_\alpha$ in the given product. The number of all those in which the total number of common operators do not form a self-conjugate subgroup of $G_\alpha$ is evidently $ap + bq$, $a$ and $b$ being positive integers. If we exclude $G_\alpha$ itself we may readily show that the number of those in which the total number of common operators is a self-conjugate subgroup of

* Frobenius, *Berliner Sitzungsberichte* (1895), p. 988.

$G_a$ is of the same form. Hence it remains only to consider the number of those which have no operator besides identity in common with $G_a$.

All these subgroups may be divided into two classes, viz: (1) those which are transformed into themselves by $G_a$, and (2) those which are transformed into different groups by the operators of $G_a$. The number of the former class may evidently be written in the form $ap + bq$, $a$ and $b$ being positive integers. If a group of the latter class occurs, all its operators must be commutative to every operator of $G_a$ * and hence $r > p(q-1)$. In this case the given theorem is evidently true. It may be observed that the number of self-conjugate subgroups of $G$ is not necessarily of the given form, *e. g.*, the direct product of two non-commutative groups of order 21 contains only two self-conjugate subgroups of this order.

CORNELL UNIVERSITY,
*February*, 1898.

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## NOTE ON THE TETRAHEDROID.

BY DR. J. I. HUTCHINSON.

(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

IN a brief paper, "A special form of a quartic surface," *Annals of Mathematics*, vol. 11, p. 158, I have called attention to an interesting special form of the locus of the vertex of a cone passing through six points. I wish to point out in this note the connection between this special surface and the tetrahedroid.

Given six arbitrary points in space 1, 2, 3, 4, 5, 6. These determine a system of $\infty^3$ quadric surfaces each of which pass through the six points. Denote this configuration by $\Sigma$.

Choose any arbitrary point $P$ and consider the polar planes of $P$ with respect to the system of quadrics. There are determined in this way $\infty^3$ planes forming a configuration $\Sigma_1$.

To a quadric in $\Sigma$ corresponds a plane in $\Sigma_1$. The vertices of the cones of $\Sigma$ have for locus a surface $K$ of the fourth order. The planes of $\Sigma_1$ corresponding to the cones of $\Sigma$ envelope a Kummer surface. The point in each plane corresponding to the cone vertex is the point of tangency.

* Dyck, *Mathematische Annalen*, vol. 22, p. 97.

To the twisted cubic $k$ determined by the six basis points corresponds a single point $O$ in $\Sigma_1$, which point is a node of the Kummer surface. To the fifteen lines 12, 13, ···, correspond 15 points (12), (13), ···. These are the remaining nodes of the Kummer surface.*

Suppose now that the six points 1, 2, ···, 6 form an involution on the cubic $k$, and that the lines 12, 34, 56, join the points that are paired in the involution. The quadric $Q$ determined by these three lines will then contain the cubic $k$, and hence the corresponding plane in $\Sigma_1$ will pass through the four nodes $O$, (12), (34), (56) of the Kummer surface.

The Kummer surface accordingly becomes in this case a tetrahedroid.†

A quadric of $\Sigma$, since it contains six given points, may be required to pass through the three lines 13, 14, 23, being thereby completely determined. In the case of involution this quadric also contains the line 24. The corresponding plane in $\Sigma_1$ contains the four nodes (13), (14), (23), (24).

Similarly, the lines 35, 45, 36, 46, lie on a quadric of $\Sigma$, to which corresponds a plane containing the nodes (35), (45), (36), (46).

Finally, a quadric containing 15, 16, 25, 26, corresponds to a plane containing (15), (16), (25), (26).

An interesting question is suggested in this connection. It is well known that the Kummer surface is determined by six arbitrary points chosen for nodes. What relation exists among these nodes when the surface becomes a tetrahedroid? The answer is, *The six nodes form an involution on the twisted cubic determined by them.* I doubt whether this result can be proved in a simple manner from the correspondence established between $\Sigma$ and $\Sigma_1$. I will limit myself here to showing analytically the existence of this geometric property for the Fresnel wave surface (the projective equivalent of the tetrahedroid).

Taking the equation of the wave surface in the usual form and introducing a fourth variable $w$ to make it homogeneous, consider, for example, the six nodes

$$(ib\beta, ia\alpha, 0, \pm\gamma), \quad (c\gamma, 0, \pm a\alpha, i\beta), \quad (\alpha, \pm\beta, \gamma, 0),$$

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*A detailed study of the correspondences between $\Sigma$ and $\Sigma_1$ is given by Reye: "Ueber Strahlensysteme zweiter Classe und die Kummer'sche Fläche," etc. (*Crelle*, vol. 86, pp. 84.)

†Cf. Cayley: "Sur un cas particulier de la surface du quatrieme ordre avec 16 points singuliers." (*Crelle*, vol. 65, p. 284.) Reye remarks (l. c., pg. 106) that the wave surface is apparently a special case (ein ziemlich specielles Beispiel) of the $\Sigma$, $\Sigma_1$ configurations, but carries his remark no further.

where

$$a = \sqrt{b^2 - c^2}, \quad \beta = \sqrt{c^2 - a^2}, \quad \gamma = \sqrt{a^2 - b^2}.$$

Transform to a new system of coördinates $x_1, x_2, x_3, x_4$, by means of the equations

$$\begin{aligned} x_1 &= (c - b)\,[a\alpha\beta x - (b\beta^2 - c\gamma^2)y + iac\alpha\gamma w], \\ x_2 &= (b + c)\,[a\alpha\beta x - (b\beta^2 + c\gamma^2)y + iac\alpha\gamma w], \\ x_3 &= (a + c)\,[a\alpha x - b\beta y + c\gamma z], \\ x_4 &= (c - a)\,[a\alpha x - b\beta y - c\gamma z]. \end{aligned}$$

With respect to this new system, the coördinates of the six chosen nodes are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), $(e_1, e_2, e_3, e_4)$, $(e_2, e_1, e_4, e_3)$, where

$$\begin{aligned} e_1 &= \beta\,(b^2 - c^2)\,(c - a), \quad e_3 = (a^2 - c^2)\,(c + b), \\ e_2 &= \beta\,(b^2 - c^2)\,(c + a), \quad e_4 = (a^2 - c^2)\,(c - b). \end{aligned}$$

These six points form an involution of the kind described. (See *Annals of Mathematics*, vol. 11, p. 159.)

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## NOTE ON INTEGRATING FACTORS.

BY MR. PAUL SAUREL.

(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

If the differential equation

$$X_1 dx_1 + X_2 dx_2 + \cdots + X_n dx_n = 0, \quad n \geqq 3, \qquad (A)$$

be integrable, and if $u =$ constant be the integral of this equation, then, as is well known, there exists a function $M$ such that

$$du \equiv MX_1 dx_1 + MX_2 dx_2 + \cdots + MX_n dx_n.$$

And as

$$\frac{\partial u}{\partial x_1} \equiv MX_1, \quad \frac{\partial u}{\partial x_2} \equiv MX_2, \cdots \frac{\partial u}{\partial x_n} \equiv MX_n,$$

and $$\frac{\partial^2 u}{\partial x_r \partial x_s} \equiv \frac{\partial^2 u}{\partial x_s \partial x_r},$$

$M$ must satisfy the conditions

$$\frac{\partial}{\partial x_r}\left(MX_s\right) \equiv \frac{\partial}{\partial x_s}\left(MX_r\right). \tag{1}$$

The object of this note is to establish the conditions for the existence of an integrating factor which shall contain but one of the variables, and to show that such an integrating factor, if it exists, is unique.

If $M$ be a function of only one of the variables, $x_1$ for example, the conditions (1) yield the $n-1$ equations

$$M\frac{\partial X_1}{\partial x_r} = M\frac{\partial X_r}{\partial x_1} + X_r\frac{dM}{dx_1}, \qquad r = 2, 3, \cdots, n;$$

or, in better form,

$$\frac{\dfrac{\partial X_1}{\partial x_r} - \dfrac{\partial X_r}{\partial x_1}}{X_r} = \frac{\dfrac{dM}{dx_1}}{M}, \quad r = 2, 3, \cdots, n; \tag{I}$$

and the $\frac{1}{2}(n-1)(n-2)$ equations

$$M\frac{\partial X_s}{\partial x_r} = M\frac{\partial X_r}{\partial x_s},$$

or $$\frac{\partial X_s}{\partial x_r} = \frac{\partial X_r}{\partial x_s}, \quad r \neq s; \quad r, s = 2, 3, \cdots, n. \tag{II}$$

In equations (I) the right-hand member is a function of $x_1$ alone, and the left-hand members are therefore all equal to the same function of $x_1$.

Thus the conditions necessary for the existence of an integrating factor which shall be a function of $x_1$ alone, are that the left-hand members of equations (I) shall be identically equal to one and the same function of $x_1$, and that equations (II) shall hold identically,—$\frac{1}{2}n(n-1)$ conditions in all.

These are also the sufficient conditions. For, if they be satisfied, put

$$\frac{\dfrac{dM}{dx_1}}{M}$$

equal to one of the equal expressions

$$\frac{\frac{\partial X_1}{\partial x_r} - \frac{\partial X_r}{\partial x_1}}{X_r}, \qquad r = 2, 3, \cdots, n,$$

and integrate; then is $M$, except as to an arbitrary constant factor, uniquely determined. And the function thus determined is an integrating factor, for the conditions (1) are all satisfied in virtue of equations (I) and (II).

If the proof given above be applied to the case of an equation containing two variables, it will show that the necessary and sufficient condition for an integrating factor containing $x_1$ alone is that

$$\frac{\frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1}}{X_2} = f_1(x_1).$$

If at the same time the condition for an integrating factor containing $x_2$ alone be satisfied, viz:

$$\frac{\frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}}{X_1} = f_2(x_2),$$

then

$$\frac{X_1}{f_1(x_1)} = -\frac{X_2}{f_2(x_2)},$$

and the differential equation may be put into the form

$$f_1(x_1)dx_1 - f_2(x_2)dx_2 = 0$$

in which the variables are separate.

The uniqueness of the integrating factor containing but one of the variables may also be obtained very simply from the known theorem that the ratio of any two integrating factors is an integral of the differential equation. For, since this ratio contains all the variables, if only one of the variables appears in the first integrating factor, the second must contain all the remaining variables, and may contain all the variables.

NEW YORK,
*January*, 1898.

# EARLY HISTORY OF GALOIS' THEORY OF EQUATIONS.

BY PROFESSOR JAMES PIERPONT.

(Read before the American Mathematical Society at the Meeting of February 26, 1897.)

THE first part of this paper will treat of Galois' relations to Lagrange, the second part will sketch the manner in which Galois' theory of equations became public. The last subject being intimately connected with Galois' life will afford me an opportunity to give some details of his tragic destiny which in more than one respect reminds one of the almost equally unhappy lot of Abel.* Indeed, both Galois and Abel from their earliest youth were strongly attracted toward algebraical theories; both believed for a time that they had solved the celebrated equation of fifth degree which for more than two centuries had baffled the efforts of the first mathematicians of the age; both, discovering their mistake, succeeded independently and unknown to each other in showing that a solution by radicals was impossible; both, hoping to gain the recognition of the Paris Academy of Sciences, presented epoch making memoirs, one of which

* Sources for Galois' Life are:

1° Revue Encyclopédique, Paris (1832), vol. 55.

*α* Travaux Mathématiques d Évariste Galois, p. 566–576. It contains Galois' letter to Chevalier, with a short introduction by one of the editors.

*β* *Ibid.* Nécrologie, Évariste Galois, p. 744–754, by Chevalier. This touching and sympathetic sketch every one should read.

2° Magasin Pittoresque. Paris (1848), vol. 16. Évariste Galois, p. 227–28. Supposed to be written by an old school comrade, M. Flaugergues, an ardent admirer of Galois.

3° Mes Mémoires, A. Dumas, (père). Paris (1869), vol. 8, p. 159–161 and p. 166–169. Gives an account of the notorious affair of the Vendanges de Bourgogne and of Galois' first trial.

4° Journal de Mathématiques, (1846), v. 11. Oeuvres Mathématiques d'Évariste Galois, p. 381–444. The avertissement is by Liouville and contains some personal notes about Galois.

5° Nouvelles Annales de Mathématiques. Paris (1849), vol. 8, p. 452. A few lines only.

6° Annales de l'École Normale. Paris (1896), 3 Ser. vol. 13, p. 197–266. La Vie d'Évariste Galois, by P. Dupuy, professor of history in the École Normale. This is the most extensive biography of Galois. It contains all that is essential in the five preceding references and an immense amount of other matter. It goes without saying that I am indebted to it for many of the details here given.

7° Oeuvres Mathématiques d'Évariste Galois. Paris, 1897. The introduction by Picard contains interesting remarks.

was lost forever, the other published first twelve years after the death of its illustrious author; in the case of both the passionate feverish blood of genius led them to a premature death, one at the age of twenty, the other at twenty-six. Both died leaving behind them theories whose thresholds they had hardly crossed, theories which, belonging to the first creations of this century, will keep their memories perennially green. I shall have no occasion in this paper to speak of anything except Galois' algebraical theories. It is, however, only just to his memory to recall a fact too universally overlooked that in spite of his extreme youth he was perfectly familiar with the researches of Abel and Jacobi and that he was certainly in possession of the most essential results of Riemann in regard to Abelian integrals discovered twenty-five years later. A remark he makes regarding a generalization of Legendre's equation connecting the periods of elliptic integrals of the first and second species

$$FE' + EF' - FF' = \frac{\pi}{2}$$

leads us to suspect that he had also anticipated some of the results of Weierstrass and Fuchs in this field. All goes to show the justness of Picard's remark* that "if only a few years more had been given him to develop his ideas in this direction he would have been the glorious continuator of Abel and would have erected in its most essential parts the theory of algebraic functions of one variable as we know it to-day."

### 1. *Galois' relations to Lagrange.*

It is well known that Galois, like Abel and Ruffini, received inspiration from the writings of Lagrange regarding the algebraic solution of equations, in particular his mémoire "Sur la résolution algébrique des équations" and his "Traité de la résolution des équations, etc.," but no one, I believe, has remarked how well Lagrange had prepared the way for Galois.

Let us capitulate rapidly the principal facts of Lagrange's theory referring for more detail to my paper† on "Lagrange's place in the theory of substitutions." In the first place he gave the theory of the solution of general equations $f(x) = 0$ of degree $n$ an entirely new basis in employing rational func-

* Introduction to the new edition of Galois' Oeuvres, which has just appeared under the auspices of the Société Mathématique de France.

† BULLETIN, 2 Series, vol 1, p. 196. (1895.)

tions of the roots of $f$ to build his resolvents. Two rational functions of the roots of $f$ invariant for the same permutations of the roots he calls similar and shows that they are roots of rational equations whose degrees are equal and a divisor of $n!$; furthermore, they are rational in one another. If $\varphi$ be a rational function of the roots taking on $r$ values for all possible permutations and $\psi$ another function which changes its value for all the permutations which alter $\varphi$ as well as for some of the permutations which leave $\varphi$ unaltered taking on in all $rs$ values, then $\varphi$ is rational in $\psi$ while $\psi$ is root of an equation of degree $s$ whose coefficients are rational in $\varphi$. This being so, a function $t$ which changes its value for every permutation enjoys the remarkable property that the roots themselves, $x_1, x_2, \cdots, x_n$, of $f$ are rational in $t$. Such a function, and Lagrange remarks that its simplest form is

$$t = ax_1 + bx_2 + \cdots + lx_n,$$

is root of a rational equation of degree $n!$, and all its roots are rational in any one of them.

This resolvent plays an important part in Lagrange's theory; I shall call it Lagrange's resolvent and denote it by $L(t) = 0$. The solution of $f = 0$ and $L(t) = 0$ are evidently equivalent problems. Let us see how a general scheme for solving the equation $f = 0$ may be deduced from these facts. Take a rational function $\varphi$ of the roots of $f$ and form its resolvent. Its solution gives its roots $\varphi_1, \varphi_2, \varphi_3, \cdots$ as known algebraic functions of the coefficients of $f$. That is, certain rational functions of the roots of $f$ are now known which at the start were not. The effect of this is to make the Lagrangian resolvent $L = 0$ which before was irreducible split up into a certain number of equal factors whose coefficients are rationally known. In other words, the determination of $t$ which before depended upon an equation of degree $n!$, depends now upon an equation of less degree say $L_1(t) = 0$. Take now another rational function $\psi$ of the roots of $f$ which is not rational in the $\varphi$'s. This will give rise to a resolvent whose coefficients are rational in the $\varphi$'s; its solution makes still other rational functions of the roots known which before were not and the effect of this is to make $L_1(t)$ reducible and thus $t$ depends upon an equation of still less degree $L_2(t) = 0$ with known coefficients. Continuing in this way, the degree of the equation upon which $t$ depends becomes less and less, at last it must depend upon an equation of first degree, and then $t$ is rational in known quantities. But since the roots of $f$ are rational in $t$, these roots are also rational in known quantities.

Now this is precisely the Galoisian scheme of solution. In other words, the task Galois had before him was: 1° to formulate the solution of *any* equation as here sketched and as indicated with considerable clearness in Lagrange's writings, and 2° to extend the theorems upon which this scheme of solution depends, from the limited case here considered, viz. equations whose coefficients are independent variables, to the case of any special equation and an arbitrary domain of rationality. The application Lagrange made of his theory to solve the special equations upon which the roots of unity of prime order depend was doubtless of great assistance to Galois in his work of generalization. Lagrange proves in this connection the cardinal fact that all cyclic functions of the roots are rationally known and uses this as the base of his solution. The cyclic group is precisely the group which here plays the same rôle as the symmetric group does for general equations.

2. *How Galois' algebraic theories became public.*

As already observed the manner in which this took place is so intimately connected with his short and agitated career that a few details of his life are indispensable.

Évariste Galois was born October 25, 1811, in the little town of Bourg-la-Reine, a few miles to the south of Paris. His father, Nicolas-Gabriel, as his grandfather before him, was proprietor of a flourishing school, established before the French Revolution and at Évariste's birth a corporate part of the University of France. His mother, Adélaïde Demante, was daughter and sister of eminent jurists of the Faculté de Droit at Paris. From her father she had received a careful and extensive education in the classics. Till Évariste was twelve years old she was his only teacher. At this stage, that is, in the autumn of 1823 he entered Louis-le-Grand, one of the celebrated lycées of Paris, a stone's throw from the Sorbonne and the Collége de France. At fifteen he first began the study of mathematics, entering the class known as *Mathématiques préparatoires*. A new world now opened up to him; the study of mathematics became the absorbing passion of his life. The Geometry of Legendre he is said to have read through as one reads a novel. Soon ordinary treatises did not satisfy him; they lacked, so he said, the stamp of the great inventors; he began the study of the original authors, especially Lagrange. In the Autumn of 1828 he entered the class of *Mathématiques spéciales*, under the charge of Professor

Richard, an excellent mathematician and a man of noble character. Under his intelligent and sympathetic direction Galois' progress became even more remarkable. His original solutions of problems proposed in the class were explained by Richard with just praise to his admiring classmates. In the trimestrial reports of the school we find Richard writing: "Cet élève a une supériorité marquée sur tous ses condisciples," and again "Cet élève ne travaille qu' aux parties supérieures des Mathématiques," while reports on his other studies are a series of lamentations caused by Galois' neglect. So, for example, this: "Il ne fait absoluement rien pour la classe. C'est la fureur des Mathematiques qui le domine." At this epoch, that is, when seventeen years old, Galois made his first discoveries in the theory of equations which can be solved by radicals. He drew up a short mémoire on the subject and Cauchy took it to present to the Academy of Sciences. The mémoire was never heard of again, although Galois reclaimed it several times at the secrétariat. At this time also his first paper, entitled "Démonstration d'un théorème sur les fractions continues périodiques" was published, appearing in the March number of the *Annales de Gergonne* for 1829.

Galois ardently desired to enter the École Polytechnique, the first school of Mathematics in France. Twice he presented himself for examination but was rejected both times to the astonishment of all who knew him. Two reasons probably lead to this deplorable mistake. Galois had the habit of working almost exclusively with his head and to deal with the broad aspects of a subject. To his remarkable mind many things seemed trivial or self-evident, which required demonstration for those less gifted, and complaint had already been made on various occasions that he was somewhat obscure in the expression of his ideas. When therefore at the examination it became necessary to work out on the blackboard, explaining as he proceeded, before a numerous audience questions of pure detail he was greatly embarrassed. He did not have what one called *l' habitude du tableau*. The second reason is that the examinators were flagrantly incapable of appreciating the extraordinary talents of the youth they had before them. At the distance of twenty years the injustice of his examinators was still keenly felt, for Terquem in a short notice on Galois which appeared in the Nouvelles Annales* writes: "Nous répéterons ici et nous ne cesserons de répéter une réflexion que nous

* Vol. 8 (1849), p. 452.

avons déjà consignée mainte fois: Un candidat d'une intelligence supérieure est perdu chez un examinateur d'une intelligence inférieure. *Barbarus hic ego sum quia non intelligor illis.* Certes M. Liouville qui nous a fait connaître le génie de Galois ne l'aurait pas jugé incapable."

Nothing was left to Galois but to enter the École Préparatoire,* a fate which filled his heart with bitterness and despair. At this crisis Galois' father, whom he dearly loved, died a violent death. As mayor of his native town, M. Galois had defended for many years the interest of the liberals against the grasping power of the clerical party. Driven at last to desperation by their incessant and venomous attacks, his kindly nature succumbed and in a moment of despondency he committed suicide. Still Galois continued with his mathematics, adding to his already vast stores of erudition and maturing his own inventions. During the first year at the École, that is, when eighteen years old, he published four mémoires. One of these is a brief summary of some of his results regarding the algebraic solution of equations; another is devoted to the theory of those numbers which we now call Galois' imaginaries and which play an important part in the theory of groups to-day. In January of this year (1830) Galois presented to the Academy another mémoire containing an account of his researches written out with care and detail. He placed great hope in this. The manuscript was given to Fourier to read, who shortly after died, and the mémoire was lost. Six months later the Revolution broke out which drove Charles X. from the throne and installed Louis Philippe in his stead. In this movement Galois entered with heart and soul. The attitude which the director of the school, M. Guigniault, took toward the new government offended Galois so deeply that he was rashly led to take part in a polemic which the *Gazette des Écoles* was waging against Guigniault. The scandal that this raised produced Galois' expulsion (Dec. 9, 1830). Powerful friends however interested themselves in him. A minister of the Royal Council, M. Barthe, summoned the unfortunate youth and bade him not to be concerned for the future. Poisson invited him to make a new redaction of his researches and volunteered to present it himself to the Academy. Inspired by fresh hopes Galois wrote the only finished mémoire we have of his grand theory of the solution of equations. It bears the title "Sur les conditions de

*This took the place of the École Normale suppressed in 1822. Dupuy speaks of it as a faint and humble copy of this celebrated school. After the revolution of 1830 it became the École Normale again.

résolubilité des équations par radicaux." At the same time he opened a course on Higher Algebra which was held in a bookstore near the Sorbonne. The *Gazette des Écoles* announced it as follows: "This course will be held Tuesdays at quarter past one; it is destined for those who, feeling how incomplete courses on algebra are, desire to penetrate more deeply into this science. Many of the theories to be given are entirely new, none have ever been developed in university lectures. We content ourselves to mention a new theory of imaginaries, a theory of equations which are soluble by radicals, the theory of numbers, and the theory of elliptic functions treated entirely by algebra." But his hopes were short lived; the mémoire intrusted to Poisson was returned, being declared unintelligible.

Already associated with the most reactionary faction of the republican party, Galois now plunged deeper and deeper into the political turmoil which was shaking France at its foundations. Twice arrested for political offenses, he spent the greater part of the last year and a half of his life in the prison of Sainte Pélagie. On being released the second time he became engaged in an affair of honor which resulted in his death (May 31, 1832). The night before the duel he spent in drawing up with feverish haste an account of his discoveries. No one can read this scientific testament written in the form of a letter to his friend and schoolmate, Auguste Chevalier, without emotion; it is one of the most touching documents in the history of science. Cut off in the very commencement of his career, conscious of the importance of his discoveries, he spent the last few hours of his life trying to save to posterity what his contemporaries had been loth to accept. He requests Chevalier to publicly beg Gauss and Jacobi to give their opinion, not on the truth but upon the importance of his theorems. "Après cela," he adds, "il y aura, j'espère, des gens qui trouveront leur profit à déchiffrer tout ce gâchis."

Chevalier was thus appointed Galois' scientific executor. Conformably to Galois' wishes the letter we have been speaking of was published the following September in the Revue Encyclopédique and an editorial note states that all Galois' manuscripts would shortly appear in the same journal under the editorship of Chevalier. This was not to be the case. Nothing more was published until 1846 when a part of them, edited by Liouville, appeared in Vol. XI. of the Journal de Mathématiques. In the preface which accompanies this collection Liouville remarks: "When, at the desire of Évariste's friends, I began an attentive study of

all his published mémoires and the manuscript he left behind, I considered it my sole task to disentangle, as far as possible, what there was new in these productions in order to make its value more evident. My zeal was soon rewarded and it gave me the greatest pleasure when, on filling some slight gaps, I recognized the entire correctness of the method whereby Galois proves this beautiful theorem: In order that an irreducible equation of prime degree be soluble by radicals it is necessary and sufficient that all its roots are rational in any two of them. This method, eminently worthy of the attention of geometers, will alone suffice to assure our compatriot a place among the few mathematicians who merit the title of inventor." Liouville's purpose was to publish all Galois' works and to add a commentary in which he intended to complete certain passages and elucidate various delicate points. Unfortunately this plan was never executed, only two new papers were published, the mémoire written at Poisson's invitation already mentioned and a fragment of a second mémoire entitled. "Des équations primitives qui sont solubles par radicaux." A note at the end of the preface informs us that the press of matter for publication as well as the extent of Galois' manuscript makes it necessary to give only a part in the current number, the rest following in the next volume. As we just said, this promise was not fulfilled.

The next reference I find to Galois' theory for the solution of algebraic equations is in a footnote, p. 344 of the first edition of Serret's Algèbre Supérieure, published in 1849: "My friend, M. Liouville," he remarks, "has announced to me his intention to publish one day some developments regarding this remarkable work. It is only by aid of these developments, a part of which M. Liouville has kindly communicated to me, that I have succeeded in comprehending certain points of Galois' Mémoire, whose study only those geometers can undertake who have occupied themselves quite specially with the theory of equations. For these reasons I have been prevented from presenting here the discoveries of Galois." The cause of Liouville's silence can only be conjectured. It seems probable that, on a more careful revision, preparatory to publishing, certain points in his demonstrations did not seem to be rigorously established. At any rate, Betti, having asked him in a paper published in the Annali of Tortolini, in 1851, not to deprive the public any longer of the results of his study, proceeded to publish his own commentary in the same journal the following year. Thus in 1852, twenty years

after Galois' death, his theory of the resolution of algebraic equations was for the first time made intelligible to the general public and established with complete rigor. A few words more on the further history of Galois' theory will complete my account. According to Weber,* Kronecker probably first became acquainted with it during his visit to Paris in 1853 where he was associated intimately with Hermite, Bertrand, and other leading French mathematicians. The first mention Kronecker makes of Galois' name is in a letter to Dirichlet in March, 1856. Dedekind also became very early acquainted with Galois' theory since it is known that he lectured in the winter of 1857–58 on higher algebra and in particular on Galois' theory. According to Weber† this was probably the first extensive account of Galois' theory given at a German university. The first account of it given in a text-book on algebra is in the third edition of Serret's algebra (1866). This, together with Jordan's classic treatise which appeared in 1870, made a knowledge of Galois' theory possible to all the world.

Perfectly just was Galois' estimate of his own discoveries when he said shortly before his death: "*J'ai fait des recherches qui arrêteront bien des savants dans les leurs.*"

YALE UNIVERSITY,
*February*, 1898.

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## LOVE'S THEORETICAL MECHANICS.

*Theoretical Mechanics, an Introductory Treatise on the Principles of Dynamics.* By A. E. H. LOVE, M.A., F.R.S., Fellow and Lecturer of St. John's College, Cambridge. Cambridge, The University Press, 1897. 8vo, xiv + 379 pp.

THIS is a text-book on dynamics intended for the useof students who have some knowledge of differential and integral calculus and coördinate geometry. The statements of first principles necessarily relate to motion in three dimensions, but the systematic development of the subject is for the most part confined to the motion in two dimensions of particles and rigid bodies. A notable feature of the book is the careful attention which is paid to the statement of the theory of dynamics, not merely as a basis for mathematical problems, but also as a branch of science. In this respect it stands in marked contrast to most other textbooks of similar scope.

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* *Mathematische Annalen*, vol. 43, p. 1.
† Algebra, vol. 1, Einleitung, p. 7.

Newton's theory is one which, partly on account of the fundamental character of it, has always been in danger of becoming a superstition, supposed to have some higher validity than the facts which it summarizes. Newton himself, though he discusses the evidence within his reach, did not indicate the position of the theory very clearly; and the discussion of the subject in Clerk Maxwell's "Matter and Motion" shows how, in this connection, science is in danger of being tinged with mysticism. Yet the correctness of the theory is very far from being obvious; and, to bring it into harmony with known facts, it requires more careful statement than it receives at the hands of most writers on the subject. One fundamental idea of it, that of the base relative to which motion is to be measured, has frequently been stated in such a form as to be unintelligible; and another point involved in it, namely the division of matter into particles, is not supposed to be in agreement with the structure of any actual body. Mr. Love has no superstitions; and, though we may not agree with him on all points, he understands how the subject should be approached in order that the theory may be put on a sound basis.

The first question which arises is that of the scientific measurement of time. It may be doubted whether Mr. Love treats this in the best way. He makes too much of our freedom to select at pleasure a standard process by means of which to measure time; a freedom which, though of course it may be claimed, has never been exercised freely. And he does not appear to appreciate the fundamental character of the notion of uniform time which is based upon the agreement of all methods of measuring time which depend upon repetitions. A repetion method is one according to which a period of time is measured by the number of times a given physical operation can be repeated in the course of it, allowance being made for variations in the conditions. There is some indication, in the Principia, that Newton had the notion of repetitions being the test of his so-called absolute time; and this notion lies at the root of all physical theory in which time is a factor. If a lump of sugar takes longer to melt in a cup of tea than a lump of the same size did yesterday, the difference has nothing to do with the fact that the two experiments are not carried on simultaneously, but is to be traced entirely to other differences in their circumstances and surroundings. Of course we can imagine this view being upset, as it would be if it were shown that the tests by two different repetition meth-

ods disagreed irreconcileably. In the meantime the theory that there is no such disagreement is so well established that it forms a possible basis for a definition of uniform time. A clock is a machine expressly designed for the performance and counting of repetitions; but the comparison of the results given by the best clocks establishes the successive rotations of the earth, relative to the stars, as being so trustworthy a series of repetitions that no artificial method can at present compete with it. Thus the rotation of the earth is, from this point of view, established as a practical standard; and we are fortunate in having so good a one, which can be referred to so easily and accurately. But the choice of it is presented to us by Mr. Love as if it were merely arbitrary. He says that it is desirable that the choice should be so made that a number of processes uncontrollable by us should be uniform, or approximately uniform; but he does not say that we are guided by the aim of making those processes most uniform in time which are, so far as our knowledge goes, most uniform in their other conditions. We come to the root of the matter when it is suggested that the adoption of a standard slightly differing from that given by the earth's rotation may be advisable, in order to enable the law of gravitation to account for one of the moon's inequalities. Mr. Love says that this may be advisable because it may be a simpler statement of the facts to change the standard of time and keep the law of gravitation than to keep the standard of time and throw over the law of gravitation. We do not wish to suggest any other standard for the choice of a system of "laws of nature" than the simplicity of the statement which it gives of the facts. But we think that Mr. Love's way of putting the case is rather misleading; for it can hardly be questioned that we should wish to save the repetition test of equal times, rather than the law of gravitation, if it should turn out that these come into conflict. Mr. Love, in his last paragraph, mentions the retardation of the earth's rotation by tidal friction, and says that it may be advisable to save the laws of energy at the expense of the uniformity of the earth's rotation relative to the stars. He might have added that, since the moments of inertia of the earth certainly undergo some change, the consideration of angular momentum shows that its rotation cannot give, with perfect accuracy, a measure of time with reference to which the laws of motion can be accurately true. It may be said that this inconsistency is so minute that it does not affect the agreement of the theory with all that we know about

the facts. All the same it is making the statement of a theory rather puzzling to give it, without warning, in a form in which it is self-contradictory. If the repetition test is adopted, the question of consistency with the law of gravitation becomes a matter of speculation, for we cannot at present apply the test accurately enough to settle it. Nevertheless we are inclined to regard this test as affording the most satisfactory basis for the definition of uniform time.

The treatment of the base (or axes) of reference, established by Newton's theory of kinetics, for the measurement of motion is on the whole good. But it is a little lacking in simplicity, in consequence perhaps of the author's desire not to countenance too much extension of the theory into the unknown. Moreover, one point in the theory is missed. No doubt the omission is intentional, but as no facts are produced in support of it, we do not see why the theory should be robbed of a point which has hitherto belonged to it. The situation is this: Newton presumably enunciates his theory for the whole universe, at any rate he does not define any limits for the dynamical system to which it is to apply. The theory is that there is one base such that the motion of bodies relative to it is that which is expressed by the "laws of motion," mass being suitably distributed. Two bases are regarded as identical from this point of view if one moves relative to the other uniformly without rotation. For the discovery of a base satisfying the theory some knowledge would be needed of the relative motions of all bodies belonging to the system. Motion relative to this base is what Newton calls "absolute" motion. But our knowledge is practically almost entirely restricted to the motions of bodies within the solar system. For this system, regarded as isolated, we can find a base which, so far as it has been tested, satisfies the theory. It is usual to assume, in the absence of any indication to the contrary, that the theory would hold for a more extended system, and accordingly that, relative to a more general base, the accelerations of all points of the solar system are approximately equal. Of course, the theory might eventually be found to break down under such extension. It is, however, convenient to regard the theory as having the more extended application, and the base employed for the solar system by itself as a provisional one. But this provisional base is regarded as unique, with the proviso mentioned above. The uniqueness is what Mr. Love omits from his statement. He approaches the question by discussing what he calls the "relativity of force." He points

out that, as acceleration has no meaning unless a base is specified relative to which it is to be reckoned, so force would appear to be in the same case. He then has to go on to say that, by virtue of the requirement of the theory that forces are to occur as interactions between pairs of particles, force does not hold quite the same position as acceleration. Any base (or frame, as he calls it) will do to refer acceleration to, but a frame of reference for force must be properly chosen. But he tells us that there is a choice among several suitable frames, and that that one should be chosen which gives to forces the simplest expression. This view may perhaps be correct, but we are not furnished with any evidence in support of it. The old established theory would be better suited for an elementary treatise.

Mr. Love has a note on the subject of the conception of matter as consisting of particles which is good and suggestive. He points out the provisional character of a theory which is dependent on particles. The importance of widening the basis of the theory to meet the requirements of physics cannot be overrated, and ought to be kept constantly in view.

Regarding the book as a whole, the arrangement strikes us as rather ponderous. There are too many provisional statements which require subsequent modification. It is tiresome not to be able to distinguish at once between those definitions and explanations which are final and those which are going to be modified. Moreover, the task sometimes put upon the reader of working backwards, and seeing how the modifications affect the intermediate argument, may be a difficult one. It is part of the scheme of the book to postpone till the end the consideration of the way in which motion must be measured in order that the system of "rational mechanics," which is built up from definitions, may have any application to real bodies. In the meantime we have the weight of a body explained to be the attraction between the body and the earth, which strikes us as being most unfortunately misleading; and we have examples involving all the apparatus of ordinary life, from railway trains to billiard balls, as if the application of the ideal system to the motion of bodies relative to the earth could be assumed without more ado. When we come to the further discussion of the subject we are told that, for the motion of a body near the earth's surface, it will be sufficient to consider the body and the earth as an independent system, and there is no reference to the remarkable and instructive exception afforded by the tides.

One of the points ignored by some writers, to which attention should be called, is the proof of the "transmissibility of force" for a rigid body. The proof given, which cannot be regarded as very easy, might have been usefully supplemented by a direct investigation of the combined effect of velocities distributed among the particles of a rigid body, and the conditions which they must satisfy. This is the most direct way of obtaining the rules for compounding the forces acting on a rigid body, though it may not be the shortest.

Turning to the more mathematical parts of the book, we are rather disappointed that there is not a greater variety of method. A text-book like this should satisfy reasonable requirements as a book of reference. Thus from this point of view, if from no other, we should have expected more prominence to be given to the expressions for velocities and accelerations in terms of coordinates referred to moving axes. One case is given, as a note in small type near the end of the book, namely, that in which the coordinates are referred to a pair of rectangular axes rotating in their own plane about the origin. Mr. Love refers in the preface to the preference shown to analytical as compared with geometrical methods, nevertheless it is a little surprising to find no use made of the hodograph for obtaining expressions for accelerations. The author has clearly made up his mind that the book should be elementary, and we find frequent evidence of the self restraint which he has practised in keeping out of it every thing of an advanced character, but a little expansion throughout would make the book much more useful. The motion of a rigid body in two dimensions appears to be well done. In this connection the term "kinetic reaction" is used, which is an improvement on the old expression "effective forces." We are inclined to grudge the large amount of space taken up by the mere enunciations of examples which are not worked. They have been collected together rather wholesale, and occupy about one hundred and thirty pages, or more than a third of the book. Indeed, taking account of the difference of type, they appear to contain nearly as much matter as the whole of the text. A more select collection would have satisfied all ordinary requirements.

On the whole, the book is both useful and interesting, and is in some respects better than any similar book in common use.

W. H. MACAULAY.

KING'S COLLEGE, CAMBRIDGE,
*January* 17, 1898.

## SCHELL'S TORTUOUS CURVES.

*Allgemeine Theorie der Curven doppelter Krümmung in rein geometrischer Darstellung.* Zur Einführung in das Studium der Curventheorie. Von Dr. Wilhelm Schell. Zweite, erweiterte Auflage. Leipzig, Teubner, 1898. 8vo, viii + 163 pp.

The first edition of Dr. Schell's Theory of tortuous curves appeared in 1859. What distinguishes this work from other treatises on the subject and makes it particularly valuable as a first introduction to the study of curves of double curvature is the simple and natural, strictly geometrical, method used throughout. The curve is here studied in itself, with the aid of the various lines and surfaces naturally connected with it, but without any reference to a system of coördinates in the ordinary sense. Barré de Saint-Venant seems to have been the first to appreciate the great power and simplicity of such a method, as appears from the third part of his classical memoir "Sur les lignes courbes non planes" (*Journal de l'École polytechnique*, vol. 18 (1845), 30$^{e}$ cahier, pp. 1–76). But while the opening remarks of this memoir show that de Saint-Venant attached no small importance to this method, it is still true that in his presentation of the theory of curves the purely geometrical treatment comes in as a kind of afterthought, as if intended merely as a means of verifying subsequently the complicated analytical calculations of the earlier parts of the paper. Besides, in de Saint-Venant's memoir the method is still far from receiving its full development.

Professor Schell's work has been known so long and so favorably that it will be sufficient to call attention to some of the additions and modifications introduced in the new edition. While the general plan of the work has remained the same, the changes are so extensive that the book has grown from 106 to 163 pages; it is revised throughout and carefully brought up to date. The insertion of headings to the sections facilitates its use for reference; but the blessing of an alphabetical index is still missing.

The introduction is almost entirely rewritten. In the first chapter of the book, after a general survey over the lines and surfaces most closely related to a tortuous curve (tangent, principal normal, binormal, rectifying line, polar axis, and the surfaces generated by these lines), the discussion of singular points, which has attracted some attention in recent years, is now illustrated by a number of figures

showing the most important singularities possible for a tortuous curve according to von Staudt's classification. While the treatment of this subject is still rather brief, the references given on p. 16 to the works of Fiedler and Wiener and to papers by Kneser, Fine, Björling, Staude, will enable the student to pursue the investigation farther.

The five following chapters (Chap. 2, on the three curvatures and their radii; Chap. 3, on the surface generated by the tangents and on the involutes of the curve; Chap. 4, on the surface generated by the polar axis and on the evolutes of the curve; Chap. 5, on the osculating sphere and cone; Chap. 6, on the surface enveloped by the rectifying plane) are not very essentially changed, although the revising hand is discernible almost on every page. The more important additions consist in the introduction of Chasles's terms "central point," "central plane" and "distributing parameter," of a generator of a scroll, in the fuller discussion of the geometrical interpretation of the radii of torsion and of total curvature, in some additional propositions about the edges of regression of the surface of the polar axes (p. 62), the rectifying surface and the surface of evolutes (p. 66–67), in the discussion of the normal sections of the rectifying surface (p. 83) and of the problem of determining curves from a given relation between their radii of curvature and torsion (pp. 83–85). The further development of the last subject, which is excluded from this work, would lead to what Professor Cesàro calls "intrinsic," and Professor Bianchi "natural," geometry of curves.

De Saint-Venant's problem concerning curves having their principal normals in common has led to so many new investigations that the next two chapters (Chap. 7, the scroll generated by the principal normals; Chap. 8, curves of constant curvature and curves with a common scroll of principal normals) had to be considerably enlarged to embody the results of the investigations of J. Bertrand, J. A. Serret, Mannheim, Fais, and others. As a tortuous curve forms, on the surface of its principal normals, an asymptotic line cutting these normals at right angles, the problem of determining how many curves on a given scroll can have this surface as the common surface of their principal normals is equivalent to the question of determining how many asymptotic lines can intersect a generator $g$ of the scroll at right angles. This form seems to have been first given to the problem by Professor Mannheim (1872). The solution is almost immediate because the scroll can, for the present purpose, be replaced by its osculating hyperboloid,

*i. e.*, by the hyperboloid determined by three successive generators $g$, $g'$, $g''$ of the scroll. The asymptotic cone of this hyperboloid being of the second order, a plane perpendicular to $g$ through its vertex will intersect the cone in 0, 1, or 2 lines; and those generators of the paraboloid which are parallel to these lines of the cone are evidently tangent to the only curves that can have $g$ as a common normal at their points of intersection with $g$. Hence, a given scroll may be the surface of principal normals for no curves, or for one curve, or for two curves. In the particular case when the osculating hyperboloid degenerates into a hyperbolic paraboloid the scroll will contain an infinite number of such curves; the scroll is in this case a common helicoid. The interesting properties of two curves having a common scroll of principal normals are proved by Professor Schell in a very simple way. Starting with Bonnet's proposition that the osculating planes of the curves at two corresponding points form a constant angle, the linear relation shown by Bertrand to exist between the curvature and torsion of either curve is deduced, the constancy of the distance between corresponding points is proved, and expressions are derived for the radii of curvature and torsion of one of the curves in terms of those of the other.

The scroll of binormals and the surface enveloped by the planes of total curvature are briefly discussed in the ninth chapter. Chapter 10, originally devoted only to the osculating helix, is enriched by the addition of an account of a conical loxodrome having with the curve a contact of the third order (pp. 122–127).

The whole of chapter 11 (pp. 128–141) is new; it applies the theory of the infinitesimal displacement of a rigid body to the trirectangular trihedral formed by the tangent, principal normal, and binormal at any point of the curve (Bianchi's principal trihedral). After deducing the indispensable lemmas from the geometry of motion here required, the author shows that the infinitesimal displacement of the trihedral, as its vertex describes an element of the curve, is a twist (or screw-motion) whose axis is the shortest distance of two successive principal normals, *i. e.*, the axis of the osculating helix, while the angle of rotation is the angle of total curvature. The translation and pitch of the twist are easily determined. A brief discussion is also given of the two ruled surfaces (sometimes called "axoids") formed by the successive positions, in space and in the trihedral, of the instantaneous axes of twist; and it is shown that the surface connected with the trihedral becomes a cylindroid if the radius of torsion of the curve is constant.

The last two chapters of the book, on cyclifying surfaces and evolutoids, have not undergone much change.

The work seems to be unusually free from serious misprints or inaccuracies. The following corrections might perhaps be worth mentioning: p. 16, footnote: Dr. Kneser's first paper appeared in Vol. XXXI (not XXI). the second will be found in Vol. XXXIV, at p. 204 (not XXIV, p. 506). of the *Mathematische Annalen*; p. 61, end of §2: the last expression for tan $\mu$ is obviously wrong, the last but one should have in the denominator $ds$ instead of $d\sigma$; p. 91: the proof of the relation $A' = B + (\frac{1}{2}\pi - a)$ seems unduly long, as the triangle $APB$ (Fig. 34), all of whose sides are infinitesimal of the same order while $P$ is a right angle, gives at once $\frac{1}{2}\pi + B$ for the exterior angle $a + A'$ at $A$; p. 110, l. 11 from foot of page: read $= 1$ for $-1$, and in the numerator of $a$ read $-1/r$ for $1/r$; p. 137, l. 15 from top: for Hauptnormalen read Binormalen.

The concluding remarks of the work give an interesting outlook on problems awaiting solution in the geometrical theory of tortuous curves. Coming as they do from one who has made a special study of the subject they will be read with great interest by all workers in this field.

ALEXANDER ZIWET.

UNIVERSITY OF MICHIGAN,
*February*, 26, 1898.

## PAGE'S DIFFERENTIAL EQUATIONS.

*Ordinary Differential Equations. An elementary text-book, with an introduction to Lie's theory of the group of one parameter.* By JAMES MORRIS PAGE, PH.D., Adjunct Professor of Mathematics, University of Virginia. The Macmillan Company, New York, 1897. 12mo, xviii + 226 pp.

This little volume is what it purports to be,—an elementary text-book with an introduction to Lie's elementary methods of integration as applied to ordinary differential equations. The contents fall into twelve chapters devoted to the following subjects in order: I. Genesis of the ordinary differential equation in two variables, pp. 1–9; II. The simultaneous system and the equivalent linear partial differential equation, pp. 10–24; III. The fundamental theorems of Lie's theory of the group of one parameter, pp. 25–59; IV. Connection between Euler's integrating factor and Lie's infinitesimal transformation, pp. 62–97; V. Geometrical

applications of the integrating factor, orthogonal trajectories and isothermal systems, pp. 100–107; VI. Differential equations of the first order but not of the first degree, singular solutions, pp. 109–118; VII. Riccati's equation and Clairaut's equation, pp. 119–131; VIII. Total differential equations of the first order and degree in three variables which are derivable from a single primitive, pp. 132–139; IX. Ordinary differential equations of the second order in two variables, pp. 140–162; X. Ordinary differential equations of the $m$th order in two variables, pp. 164–174; XI. The general linear differential equation in two variables, pp. 175–189; XII. Methods for the integration of the simultaneous system, pp. 201–211.

The author finds himself under the usual obligations which the maker of an elementary text-book has to face; these he acknowledges in the preface with a care that ought to satisfy the most scrupulous, but throughout the text one misses a number of references to Lie's lectures on differential equations and to the treatises of Boole and Forsyth which would have been helpful to both student and instructor in their supplementary reading. The few direct references to Lie's early theorems are the citations to the original memoirs in the Verhandlungen of the Christiania Scientific Society as given in Lie's published lectures; the latter are accessible to American readers while the former are not; double references might not have been amiss in these cases.

The arrangement of the material is a natural one and the exposition is clear. A detailed account of the subject matter is hardly called for here. That which is of most interest is the portion of the work devoted to Lie's theory of the group of one parameter. This part of the work, with the exception of the treatment of singular solutions mentioned below, is drawn from Lie's lectures on differential equations.* In this section of the book the author follows Lie not only in matter but also in method; this faithfulness to the original enhances the value of the book to the student who desires to go more deeply into Lie's theory of continuous groups. The presentation is largely geometrical. Numerous illustrative examples are interspersed throughout the text. Here it might be mentioned that each chapter is followed by a list of problems; answers to these are collected together at the end of the book just

* Lie: Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, bearbeitet und herausgegeben von Scheffers, Leipzig, 1891.

before a well-ordered index. The material due exclusively to Lie is that of chapters III–V and IX–XII inclusive. The theorems and processes made use of in these chapters were discussed somewhat at length in a recent review of Lie's differential equations in the BULLETIN.*

It might have been well to give the complete proof of Lie's fundamental theorem relative to the equivalence of the notions infinitesimal transformation and one parameter group. The proof requires an elementary theorem of the theory of functions, but the student can well afford to make the acquaintance of this theorem at the stage when he is reading this book. The existence theorem is assumed and this other more elementary theorem of the theory of functions might have been assumed in the same connection. This fundamental equivalence theorem is left half demonstrated when it might have been put in a form thoroughly intelligible to beginners.

The notion lineal element in the sense used by Lie is of course introduced, but one questions why the notion element association of lineal elements is not brought into play. The latter is one of the most useful of the geometrical notions of Lie and its rôle could have been indicated on a single page.

The value of the book would have been largely increased had the author given footnotes indicative of the historical relations of the various subjects studied. Footnotes of another kind would also have been especially helpful, namely those indicating limitations, extensions and connections. For example, the student learns that a differential equation of the first order admits of an indefinite number of infinitesimal transformations, but in the study of differential equations of the second order no reference is made to the corresponding theorems: 1° that an ordinary differential equation of the second order admits at most of eight independent infinitesimal point transformations; 2° that every differential equation of the second order can be transformed into every other differential equation of that order by contact transformations. In fact, the student is given no hint as to the existence of a contact transformation, not even in the most opportune place when finding what is technically known as the extension of a point transformation. These extended point transformations are the most special form of contact transformations; five lines would have sufficed to introduce the general notion at this point. On the other

*Second series, vol. IV., No. 4, January, 1898. Lie's Differential Equations, pp. 155–167.

hand an admirable and noteworthy feature of Page's book is that no opportunity is lost to indicate where it is possible to extend given theorems in two and three variables to $n$ variables.

No one detail of the subject of ordinary differential equations has been the victim of such unsatisfactory and cumbersome treatment as that of singular solutions. The method of the book before us offers relief. In three pages the author has developed a very neat method of finding the singular solution of an ordinary differential equation when such solutions exist. This method for determining singular solutions is a beautiful one from a theoretical standpoint, but its practical value is impaired because of the fact that the problem of finding the group of which a given differential equation admits is itself a problem of integration not capable of general solution. The author fails to call attention to certain elementary properties of singular solutions which the method may be made to yield in a simple manner. Of these, for example, are 1° every singular solution satisfies an infinite number of differential equations; 2° the existence of a singular solution depends upon the form of the equation. *

The translation of Lie's terminology has found in most cases fortunate English equivalents. The spirit of the original is certainly preserved in adopting the term pathcurve for Bahncurve, to name a single instance. Differential equations of the different orders which are invariant under known one parameter groups are designated as Lie's differential equations. This is a graceful way of recognizing permanently the creator of the theory of continuous groups; but with greater propriety the groups themselves could have been called Lie groups and this too in greater harmony with existing designations, putting the Lie groups of the transformation theory of differential equations alongside the Galois groups of the substitution theory of algebraic equations.

Page's book is intended for beginners. It is a pioneer in English. Its author was one of Lie's earliest pupils. Its object is a worthy one and merits the appreciation of American students of mathematics. The technicalities touched

* It may be of interest to note here that Taylor seems to have been the first to observe that a differential equation may have solutions not comprised in the general integral. He remarked this in 1715 by finding that $y^2 = 1 + x^2$ is a singular solution of $y^2 - 2xyy' + (1 + x^2)y'^2 - 1 = 0$. See pp. 270 and 360 of Frenet's Recueil d'exercices sur le calcul infinitesimal, quatrième édition, 1882.

upon here with a critical pen are to be regarded as trivial when compared with the work as a whole. In the hands of a teacher who will supplement the exercises of the text with more of those illustrative of the older methods on the one hand, and who will drill his students vigorously in reckoning with Lie's fundamental operator, the infinitesimal transformation, until they acquire facility in forming commutators,* on the other, the book is fully capable of realizing its double object of introducing the beginner† to two of the widest and most fruitful domains of mathematics—the theory of continuous groups and the theory of differential equations.

The mechanical make up of the book is up to the standard of the Macmillan press. A few trivial typographical errors will not escape the careful reader; the most flagrant probably is the occurrence of three in two consecutive lines of the preface.

The list of mathematical text-books in English is being continually and wisely augmented by the publications of The Macmillan Company, but with all that is good in the new, American students would welcome re-edited reprints of the classic volumes of Boole in the field of differential equations.

EDGAR ODELL LOVETT.

PRINCETON, N. J.,
24 *February*, 1898.

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## SHORTER NOTICES.

*Annuaire pour l' An 1898, publié par le Bureau des Longitudes.* Paris, Gauthier-Villars et Fils.

One or two alterations which have been made in the *Annuaire* for the current year call for mention. A few of the more technical astronomical details concerning variable and double stars and small planets have been omitted, while the

* This term has been suggested as the equivalent of Lie's *Klammerausdruck* to represent the operation $(U_1U_2)f \equiv U_1(U_2f) - U_2(U_1f)$, where $U_1f$ and $U_2f$ are two infinitesimal transformations; its use would have spared numerous circumlocutions in the text.

† The reader will observe that in the above opinion the writer takes exception to one advanced in *Nature* (current volume, 10 February, 1898), where the usefulness of Page's book as an introductory volume for beginners is questioned. No text-book can supply both text and teacher. In the hands of an instructor who is alive to both sides of the subject the book is susceptible of successful application to the needs of those studying the subject for the first time.

elements of those periodic comets which have been observed during part of a single revolution only, have been transferred to the *Connaissance des Temps* for 1900. New magnetic charts and elements for France have been inserted and the scientific information generally is brought up to date.

There are five appendices. MM. Loewy and Puiseux write on the progress made by the aid of photography in our knowledge of the surface of the moon; M. Poincaré considers the stability of the solar system; M. Cornu describes the scientific achievements of Fizeau; M. Janssen records the work done at the Mont Blanc Observatory during 1897; and the orations delivered by MM. Janssen and Loewy at the 50th anniversary of M. Faye's admission to the Academy are given in full. As usual these notices are clearly and simply written and thoroughly interesting, but only one calls for special remark here, namely, that of M. Poincaré.

M. Poincaré points that all the efforts made to demonstrate the stability of the solar system have been of the nature of successive approximations, the demonstrations being constructed for fictitious planets whose motions approach more or less closely to those of the solar system. It is doubtful, however, even if we restrict the problem by neglecting tidal action and the effects that are possibly produced by a resisting medium, whether the system would be really stable. The mathematical expressions for the motions show that the latter are periodic in their nature, but it is not known for what extent of time these expressions will be available. M. Poincaré draws the distinction, necessary in this connection, between a very long time and an infinite time. The results show that the motions are stable in the former case, if not in the latter. But when we consider the parts played by these other forces, the ultimate stability seems doubtful. The tides alone, whether liquid or elastic, will tend to make the whole system rotate as one rigid body, but the presence of a resisting medium, however slight—and one can hardly doubt that such a medium exists—will continually accelerate the mean motions and diminish the mean distances. One by one the planets will fall into the sun and the final state will be a single body at rest relatively to the medium which surrounds it.

ERNEST W. BROWN.

*Erkenntnistheoretische Grundzüge der Naturwissenschaften.* Von Professor Dr. P. VOLKMANN. Leipzig, Teubner, 1896. 8vo, xii + 181 pp.

For centuries the study of the various forms of knowledge was not differentiated. When the division was made the natural tendency was to go to an extreme. It must be acknowledged that the history of philosophy, in particular metaphysics, is a propædeutic to the study of all science. Until quite modern times mathematical reasoning was in nearly every case bound up with physical reasoning and it is only in most recent days that the idea has prevailed that mathematics forms a subject which is more intimately connected with logic than with other divisions of knowledge. From the connections with physics and logic it is obvious that the terminology of mathematics can only be understood by one with at least a certain knowledge of the methods of reasoning of the natural sciences and of the connotation of the words used in those sciences. This is so well understood that the language of several of the leading mathematicians of Germany and France has been enriched by words drawn from mechanics. The relation of mathematics to both metaphysics and physics justifies the mathematician in taking an interest in the book in hand, as in fact in every book of the kind since the time of Mill.

Dr. Volkmann, professor of mathematical physics at the University of Königsberg, is particularly qualified to set forth the principles of the middle ground between science and philosophy. Two quotations will indicate the scope of the book: "The discussion here is directed less to colleagues in one department than to the cultured public with scientific interests." "We may be the best logicians without progressing a single step in knowledge; possibly exactly this wish to be such good logicians prevents us from making progress in knowledge. The difficulty of a cognition does not at all lie in the difficulty of correct reasoning, it lies rather in the mode and manner of acquiring and sifting truly appropriate material, it lies also in the management of the inflexible material with which our praised and omnipresent logic has to work."

In a terse and interesting way Professor Volkmann leads up to the treatment of induction and deduction. He then takes up what he calls "Isolation" and "Superposition." Each phenomenon of nature is a composite made up of elements. The ability to isolate these elements so that they may be studied separately, and then the ability to

combine these elements properly, are necessary steps in all study of phenomena. These two processes are the products of what may be called a refined form of Mill's method of concomitant variations.

The chapter dealing with the idea of order of magnitude and measurement is of particular interest to the scientist.

H. D. THOMPSON.

## NOTES.

PROFESSOR FELIX KLEIN'S Lectures on Mathematics, known as "The Evanston Colloquium" (New York, Macmillan, 1894), are out of print. A. Hermann, of Paris, has just brought out a French translation of this work, prepared by M. L. Laugel; the title is, "Conférences sur les mathématiques, faites au congrès de mathématiques tenu à l' occasion de l' exposition de Chicago." M. Laugel has added seventeen pages of valuable bibliographical notes, thus bringing the references up to date.

M. Laugel has also accomplished the difficult task of translating into French the collected mathematical works of Bernhard Riemann. This translation is now in the press of MM. Gauthier-Villars and will soon be ready.

THE memoir on the analytical representation of direction, etc., published by Caspar Wessel, in 1797, which contains the earliest systematic attempt at a geometrical interpretation of imaginaries was the subject of Professor W. W. Beman's vice-presidential address before Section A of the American Association for the Advancement of Science last summer. The address was published in full in *Science*, vol. 6, pp. 297–307 (Aug. 27, 1897).

AT the meeting of the Paris Academy of Sciences, February 7, 1898, the death of the well-known publisher, M. JEAN ALBERT GAUTHIER-VILLARS, was announced, and resolutions were adopted in recognition of his manifold services to science, particularly of his generous assistance to the French Government and the Academy in publishing the complete works of Lagrange, Fermat, Fourier, and Cauchy.

THE Schubert prize of the Imperial Academy of Sciences, St. Petersburg, has been awarded to PROFESSOR SIMON NEWCOMB for work in theoretical astronomy.

PROFESSOR D. E. SMITH, professor of mathematics in the Michigan State Normal College, has been appointed to the principalship of the State Normal School at Brockport, New York.

DR. EDGAR ODELL LOVETT, of Princeton University, has been promoted to an assistant professorship in the John C. Green School of Science of that institution.

PROFESSORS FELIX KLEIN and GEORGE SALMON have been elected foreign members of the Belgian Academy of Sciences.

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## NEW PUBLICATIONS.

### I. HIGHER MATHEMATICS.

BALL (W. W. R.). Récréations et problèmes mathématiques des temps anciens et modernes. Traduit sur la troisième édition anglaise, par J. Fitz-Patrick. Paris, Hermann, 1897. 8vo. 4 and 352 pp. Fr. 9.00

BENEDETTI (G.). Scienza dei triangoli Pitogorici dipendenti dai tre mezzi determinati dalle proporzioni geometriche discrete. Trieste, 1897. 8vo. 21 pp. Mk. 1.80

BORTKEWITSCH (L. v.). Das Gesetz der kleinen Zahlen. Leipzig, Teubner, 1898. 8vo. 7 and 52 pp. Mk. 2.00

CURTZE (M.). See PETRI PHILOMENI, etc.

CZUBER (E.). Vorlesungen über Differential- und Integral-Rechnung. Vol 1. Leipzig, Teubner, 1898. 8vo. 13 and 526 pp. Mk. 12.00

DESAINT (L.). Sur quelques points de la théorie des fonctions. Paris, 1897. 4to. 79 pp.

DICKSON (L. E.). The quadratic Cremona transformation. (*California Academy of Sciences*, Vol. 1, No. 2.) San Francisco, 1898. 8vo. 11 pp.

DICKSTEIN (S.). See PASCAL (E.), PEANO (G.), and SCHLEGEL (V.).

ENRIQUES (F.). Lezioni di geometria proiettiva. Bologna, Zanichelli, 1898. 8vo. 9 and 379 pp. Fr. 9.00

FITZ-PATRICK (J.). See BALL (W. W. R.).

GALDEANO (DE). Las modernas generalisaciones expresadas por el álgebra simbólica. Madrid, 1896. 142 pp. Fr. 2.00

GLÖSEL (K.). Ueber Combinationen zu bestimmten Summen. Bielitz, 1897. 8vo. 32 pp. Mk. 1.50

GOURSAT (E.). Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes. Vol. II: La méthode de Laplace; les systèmes en involution; la méthode de M. Darboux; les équations de la première classe; transformations des équations du second ordre; généralisations diverses. Paris, Hermann, 1898. 8vo. 344 pp. Fr. 10.50

GRAF (J. H.). Der Mathematiker Jacob Steiner von Utzendorf. Ein Lebensbild. Bern, Wyss, 1897. 8vo. 3 and 54 pp., portrait and facsimile. Mk. 1.20

GRIECO (M.). Sulle relazioni biquadratiche fra le funzioni theta e sopra le corrispondente formole di addizione. Roma, 1897. 4to. 20 pp. Mk. 2.00

HASKELL (M. W.). On rational quadratic transformations. (*California Academy of Sciences*, Vol. 1, No. 2). San Francisco, 1898. 8vo. 12 pp.

———. On curvilinear asymptotes. (*California Academy of Sciences*, Vol. 1, No. 2.) San Francisco, 1898. 8vo. 4 pp.

IGURBIDE (J. F.). La nueva ciencia geométrica. Vol. I: Geometría del circulo. Barcelona, 1897. 4to. 386 pp. Fr. 12.00

LA VALLÉE-POUSSIN (C. DE.). Recherches analytiques sur la théorie des nombres premiers. Paris, 1897. 8vo. Fr. 10.50

LAUGEL (L.). See SCHWARZ (H. A.).

LUCAS (E.).—Récréations mathématiques. Vol. IV: Le calendrier perpétuel; l'arithmétique en boules; l'arithmétique en bâtons; les mérelles au 13e siècle; les carrés magiques de Fermat; les réseaux et les dominos; les régions et les quatre couleurs; la machine à marcher. Paris, 1898. Fr. 7.50

MACÉ DE LÉPINAY (A.). Compléments d'algèbre et notions de géométrie analytique; à l'usage des élèves de mathématiques élémentaires et de première-sciences se destinant aux mathématiques spéciales, et des jeunes filles visant à l'agrégation. 4e édition. Paris, Nony, 1898. 8vo. 486 pp.

MOORE (C. H.). Julius Firmicus Maternus, der Heide und der Christ. München, 1897. 8vo. 51 pp.

OVIDIO (E. D'). Geometria analitica. Torino, Bocca, 1898. 8vo. Fr. 10.00

PASCAL (E.). Repertorio di matematiche superiori; definizioni, formole, teoremi, cenni bibliografici. Vol. I: Analisi. Milano, Hoepli, 1898. 16mo. 15 and 642 pp. Fr. 6.00

———. Infinitesimal calculus, translated by S. Dickstein. (Polish.) (In 3 parts.) Parts I and II: Differential and integral calculus. Warsaw, 1896. 265 and 240 pp. Mk. 12.00

———. I determinanti. Milano, Hoepli, 1897. 330 pp. Fr. 3.00

PASQUALE (V. DE). Note all'algebra complementare di Pincherle. Parte I: Analisi algebrica. Messina, 1897. 8vo. Fr. 2.00

PEANO (G.). Sketch of the geometrical calculus. Translation by S. Dickstein. (Polish.) Warsaw, 1897. 8vo. 28 pp. Mk. 1.50

PETRI PHILOMENI de Dacia in algorismum vulgarem Ioannis de Sacrobosco commentarius. Una cum algorismo ipso edidit et præfatus est M. Curtze. Sumptibus societatis regiæ scientiarum danicæ. Kopenhagen, Höst, 1898. 8vo. 19 and 92 pp. Mk. 2.25

RIPERT (L.). La dualité et l'homographie dans le triangle et le tétraèdre. Paris, Gauthier-Villars, 1898. 8vo. 60 pp.

RUCKDESCHEL (F.). Transformationen in Linien- und Ebenencoordinaten nach neueren Systemen. Marburg, 1897. 8vo. 74 pp. Mk. 1.80

SHAPOSHNIKOV (A. L.). Elements of mathematical analysis, from lectures at the School of Technology. Vol. I, part I: Algebra and analysis of variables. (Russian.) Moscow, 1897. 8vo. 150 pp. Mk. 3.00

SCHLEGEL (V.). Grassmann's theory of extension; a contribution to the history of the mathematics of the last 50 years. Translation by S. Dickstein. (Polish.) Warsaw, 1896. 8vo. 51 pp. Mk. 2.50

SCHWARZ (H. A.). Sur certains problèmes de représentation conforme. Traduit avec l'autorisation de l'auteur par L. Laugel. (*Nouvelles Annales de Mathématiques*, 3e serie, t. XVI, 1897.) 8vo. 32 pp.

VASHCHENKO-ZAKHARCHENKO (M. E.). Projective geometry. (Russian.) Kiev, 1897. 8vo. 385 pp. Mk. 10.00

VILLAFANE Y VINALO (J. M.). Elementos de las teorías coördinatorias y de las determinantes con sus principales aplicaciones. 3a edicion. Barcelona, 1897. 4to. 411 pp. Fr. 13.50

VIVANTE. Corso di calcolo infinitesimale. Messina, 1897. 8vo. Fr. 8.00

## II. ELEMENTARY MATHEMATICS.

BERTRAND (G.). Trattato di algebra elementare. Traduzione italiana con note ed aggiunte da E. Betti. Nuova edizione per cura di A. Socci. Firenze, 1898. 12mo. 6 and 531 pp. Fr. 3.00

BETTI (E.). See BERTRAND (G.).

BIELAVSKY (S. I.). Collection of examples in algebra, geometry, trigonometry and physics. (Russian.) St. Petersburg, 1897. 8vo. 56 pp. Mk. 3.50

BUSSLER (F.). Mathematisches Uebungsbuch. Teil 2: Für den Gebrauch in den oberen Klassen höherer Lehranstalten (Obersekunda und Prima) zusammengestellt. 2te Auflage. Dresden, Ehlermann, 1898. 8vo. 3 and 206 pp. Cloth. Mk. 1.80

CHEMOLOSOV (S. S.). Plane trigonometry. (Russian.) Lubny, 1897. 8vo. 105 pp. Mk 2.50

COLEMAN (S. E.). An algebraic arithmetic, being an exposition of the theory and practice of advanced arithmetic, based on the algebraic equation. New York, Macmillan, 1898. 12mo. 13 and 151 pp. Cloth. $0.60

FILOGOMO (U. G.). Misurazione della circonferenza e del circolo e conseguenze. Palermo, Meccio, 1897. 8vo. 6 pp., 2 plates.

HEIS (E.). Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra. 3te Auflage. Köln, 1897. 8vo. 4 and 421 pp. Mk. 3.00

JOHANSON (F. A.). Course of trigonometry. (Russian.) Kiev, 1897. 12mo. 37 pp. Mk. 2.00

KISELEV (A.). Elementary algebra. 8th edition. (Russian.) Moscow, 1897. 8vo. 320 pp. Mk. 3.00

———. Short treatise of algebra. 2d edition. (Russian.) Moscow, 1897. 8vo. 187 pp. Mk. 2.50

LAUNAY (L.). Arithmétique, contenant plus de 800 questions et problèmes. Paris, 1897. 8vo. Cloth. Fr. 4.00

LÖWENSTERN (L. A.). Systematic solutions of the problems in algebra, geometry and trigonometry, set at the examinations for admission to the Schools of Mines, Engineering and Technology. (Russian.) Moscow, 1897. 8vo. 209 pp. Mk. 7.00

MANUEL du baccalauréat de l'enseignement secondaire classique (classes de rhétorique et de philosophie). Mathématiques. 2e édition. Paris, Nony, 1898. 16mo. 67 pp.

MENZL (O.). Ueber den mathematischen Unterricht an den höheren Handelsschulen. Reichenberg, 1897. 8vo. 43 pp. Mk. 1.50

MERCHINSKY (A.). Elementary geometry. (Russian.) Moscow, 1897. 8vo. 150 pp. Mk. 3.20

MEYER. Laerobog i Trigonometri. Kopenhagen, Lehmann og Stage, 1897.

MOUGIN (E.). Nouvelles tables de logarithmes à cinq décimales pour les nombres et les lignes trigonométriques. Blois, Migault, 1898. 4to. 38 pp. Fr. 1.25

SHMULEVICH (P.). Collection of problems in trigonometry for engineers. 618 problems with detailed solutions. 2d edition. (Russian.) St. Petersburg, 1897. 8vo. 155 pp., 1 table. Mk. 5.00

SIMON (P.). Géométrie élémentaire, à l'usage des classes de lettres. Partie I: Géométrie plane. Saumur, Verneau, 1897. (Autogr.). 8vo. 188 pp.

SOCCI (A.). See BERTRAND (G.).

THOMSON (W.). Algebra for schools. London, Chambers, 1898. 8vo. 560 pp. 4s. 6d.

TOBEL (E. VON). Geometrie für Sekundarschulen. Weitere Ausführungen für die Hand des Lehrers. Zürich, Füssli, 1898. 8vo. 7 and 100 pp. Boards. Mk. 1.80

## III. APPLIED MATHEMATICS.

ARIÈS (E.). Thermodynamique des systèmes homogènes. Paris, Gauthier-Villars [1898]. 16mo. 175 pp. Fr. 2.50

BINDER (W.). Die cotierte Darstellung auf einer Bildebene nebst einem Vorschlage: Zur einheitlichen Bezeichnung in der darstellenden Geometrie. [Progr.] Wiener-Neustadt, 1897. 8vo. 29 pp., 2 plates.

BRIGGS (W.) and BRYAN (G. H.). Advanced mechanics. Vol. I: Tutorial dynamics, with the questions of the last 13 years set at the advanced examinations of the Science and Art Department. (Organised science series.) London, Clive, 1898. 8vo. 336 pp. 3s. 6d.

DESCRIPTIVE and solid geometry and graphic arithmetic (elementary stage). Solutions to 150 questions taken from departmental examinations and Moffat's test papers. London, Moffat, 1898. 8vo. 1s. 6d.

FÖPPL (A.). Vorlesungen über technische Mechanik. Vol. III: Festigkeitslehre. Leipzig, Teubner, 1898. 8vo. 16 and 472 pp. Cloth. Mk. 12.00

FRIED (H.). Zur Theorie der Flüssigkeitsstrahlen. [Diss.] München, 1897. 8vo. 34 pp.

GILLESPIE (W. M.). Treatise on surveying, edited by C. Staley. New edition. (In 2 parts.) Part 2: Higher surveying. New York, Appleton, 1898. 8vo. Cloth. $2.50

GUTHE (K. E.). Measurement of self-inductance by alternating current and electrodynamometer. (*American Journal of Science*, Vol. 5.) 8vo. Pp. 141–143.

HAMMER (E.). Der logarithmische Rechenschieber und sein Gebrauch. Eine elementare Anleitung zur Verwendung des Instruments für Studirende und für Praktiker. Stuttgart, Metzler, 1898. 8vo. 60 pp. Mk. 0.40

HELLMANN (G.). Neudrucke von Schriften und Karten über Meteorologie und Erdmagnetismus. No. 10: Rara magnetica. 1269–1599. 10 teilweise in Facsimile wiedergegebene Schriften von Peregrinus de Maricourt, F. Falero, P. Nunes, J. de Castro, G. Hartmann, M. Cortes, G. Mercator, R. Norman, W. Borough, S. Stevin. Mit einer Einleitung. Berlin, Asher, 1898. 4to. 25, 12, 6, 2, 16, 2, 2, 2, 13, 23, 59, and 8 pp., 1 facsimile, 4 photolith. Mk. 15.00

———. No. 11: J. H. Winkler, B. Franklin, T. F. Dalibard, L. G. Le Monnier: Ueber Luftelektricität. 1746–1753. Mit einer Einleitung. Berlin, Asher, 1898. 4to. 8, 10, 16, 6, and 9 pp. Mk. 3.50

HOPPE (O.). Elementarer praktischer Leitfaden der Elektrotechnik in technisch-wissenschaftlichem Zusammenhange mit der Maschinen-, Berg- und Hütten-Technik, aufgebaut auf der technischen Mechanik als der gemeinsamen Grundlage für das Gesamtgebiet der Technik und der erklärenden Naturwissenschaften für Techniker und Nichttechniker. Essen, Baedeker, 1898. 8vo. 14 and 175 pp. Cloth. Mk. 4.00

KLEIN (F.). See ROUTH (E. J.).

MÜLLER (C. A.). Multiplikations-Tabeller, äfven användbara för Division, efter en ny plan utgifna. Autoriserad svensk upplaga. Lund, 1898. 8vo. 201 pp. Cloth. Mk. 3.50

ORAM (H. J.). See SENNETT (R.).

PECK (H. A.). The equinox and obliquity of the ecliptic for 1885,5 and the constant of the lunar equation. [Diss.] Strassburg, 1897. 4to. 45 pp.

PELLAT (H.). Cours de physique; à l'usage des élèves de la classe de mathématiques spéciales. 2e édition. (En 2 volumes.) Vol. I: Optique géométrique, mécanique, unités, instruments de mesure, 6 and 596 pp. Vol. II: Pesanteur, hydrostatique, capillarité, pneumatique, hydrodynamique, chaleur, 871 pp. Paris, Dupont, 1898. 8vo. Fr. 26.00

REED (J. O.). Ueber den Einfluss der Temperatur auf die Berechnung und Dispersion einiger Gläser und Krystalle. [Diss.] Jena, 1897. 8vo. 52 pp., 1 plate.

ROUTH (E. J.). Die Dynamik der Systeme starrer Körper. In zwei Bänden mit zahlreichen Beispielen. Autorisierte deutsche Ausgabe von A. Schepp. Mit einem Vorwort von F. Klein. Vol. I: Die Elemente. Leipzig, Teubner, 1897. 8vo. 12 and 472 pp. Cloth. Mk. 10.00

SCHEPP (A.). See ROUTH (E. J.).

SCHUMANN (H.). Einführung in die neuere Elektricitätslehre in elementar mathematischer Behandlung. Für höhere Schulen sowie zum Studium für angehende Elektrotechniker. München, Wolff, 1898. 8vo. 8 and 215 pp. Mk. 3.20

SENNETT (R.) and ORAM (H. J.). The marine steam-engine; a treatise for engineering students, young engineers and officers of the Royal Navy and mercantile marine. 3d edition. London and New York, Longmans, 1898. 8vo. 8 and 519 pp. Cloth. $6.00

SLICHTER (C. S.). Note on the pressure within the earth. (*Journal of Geology*, Vol. VI, 1898.) 8vo. pp. 65–78.

———. Harmonic curves of three frequencies. (Transactions of the Wisconsin Academy, Vol. XI, 1897.) 8vo. Pp. 449–451, plates 43–50.

SPANTON (J. H.). Complete perspective course, comprising the elementary and advanced stages of perspective, the projection of shadows and reflections, with exercises in theory and practice, also the practical applications of perspective. London and New York, Macmillan, 1898. 8vo. 18 and 282 pp. Cloth. $2.60

STALEY (C.). See GILLESPIE (W. M.).

STOLLE (R.). See WEICKERT (A.).

TAPLA (J.). Vademecum der darstellenden Geometrie. Für Schüler gewerblicher Lehranstalten, für Schüler und Absolventen des Gymnasiums, sowie für Praktiker. Wien, Fromme, 1898. 16mo. 13 and 163 pp., 39 plates. Boards. Mk. 5.00

VESELOVSKY (N. N.). Elementary theory of errors of observation and of the so-called method of least squares, with applications to geodesy. (Russian.) Moscow, 1897. 8vo. 239 pp. Mk. 7.00

WEICKERT (A.) und STOLLE (R.). Praktisches Maschinenrechnen. Eine Zusammenstellung der wichtigsten Erfahrungswerte aus der allgemeinen und angewandten Mechanik in ihrer Anwendung auf den praktischen Maschinenbau. Erläutert durch zahlreiche der Praxis entnommene Beispiele und eingeleitet durch eine leichtfassliche Darstellung der für Maschinenbauer unentbehrlichen Gesetze des allgemeinen Buchstabenrechnens. 3te Auflage. Berlin, Polytechnische Buchhandlung, 1898. 8vo. 7 and 262 pp. Mk. 3.50

WINTER (C.). Dioptrische Bilder von sehr kleinen paraxialen Objecten in centrierten sphärischen Flächensystemen. [Diss.] Rostock, 1897. 8vo. 30 pp., 1 plate.

ZEISSIG (C.). Ein einfacher Fall der transversalen Schwingung einer rechteckigen elastischen Platte. Göttingen, 1897. 8vo. 52 pp., 2 plates. Mk. 2.00